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**THE BOREL HIERARCHY AND
THE PROJECTIVE HIERARCHY
IN INTUITIONISTIC MATHEMATICS**

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The Borel Hierarchy and the Projective Hierarchy in intuitionistic mathematics

In memoriam magistri cari
Johan J. de Iongh (1915–1999)

Μοι ἔφανε βαθὸς τι ἔχειν παντάπασι γενναῖον

To me he seemed to have a kind of depth, a wholly noble one

Plato, Theaet. 183e

Ἄπ’ ὅσα ἔκαμα κι ἅπ’ ὅσα εἶπα νὰ μὴ ζητήσουνε νὰ βροῦν ποιὸς ἦμουν

From all I did and all I said let them not seek and find out how I was

C.P. Cavafy, KPYMMENA (Hidden Things), 1908

In intuitionistic analysis one may prove, using Brouwer’s Continuity Principle and an Axiom of Countable Choice, that the positively Borel sets form a really growing hierarchy. The Continuity Principle implies also that the Borel Hierarchy has a remarkable fine structure. A strong form of the Principle brings about the collapse of the Projective Hierarchy.

0 Introduction

0.1 E. Borel, H. Lebesgue, R. Baire, N. Lusin, A. Souslin and others who studied Borel sets and later also analytic and projective sets, did so partly for philosophical reasons. They found it difficult to attach a sense to the expression: “*all* subsets of the set \mathbb{R} of real numbers” and decided to focus attention on certain classes of *definable* subsets of the continuum. L.E.J. Brouwer, when developing his intuitionistic mathematics, had similar but also more fundamental concerns. He came to see that the meaning of mathematical statements is only to be found in their constructive content. Mathematicians should acknowledge this and adapt and refine their language. Further reflection on how to understand the continuum then led him to enunciate some

revolutionary axioms. We want to find out what becomes of the questions raised by Borel and Lebesgue if one takes Brouwer's point of view.

0.2 A subset of the set \mathbb{R} of real numbers will be called *positively Borel* if and only if it is obtained from open subsets of \mathbb{R} by the repeated use of the operations of countable intersection and countable union.

Given any sequence K_0, K_1, K_2, \dots of classes of subsets of \mathbb{R} , we let its *product* $\prod_{n \in \mathbb{N}} K_n$ be the class consisting of all sets of the form $\bigcap_{i \in \mathbb{N}} X_i$ where each set X_i belongs to some K_n , and we let its *sum* $\sum_{n \in \mathbb{N}} K_n$ be the class consisting of all sets of the form $\bigcup_{i \in \mathbb{N}} X_i$ where each set X_i belongs to some K_n .

We call a class of subsets of \mathbb{R} a *canonical class (of positively Borel subsets)* if and only if it is obtained from the class of the open subsets of \mathbb{R} and the class of the closed subsets of \mathbb{R} by the repeated use of the operations of countable product and countable sum.

The Borel Hierarchy Theorem proved by Borel and Lebesgue is the statement that no canonical class exhausts the collection of all positively Borel subsets of \mathbb{R} .

0.3 We do not include the operation of taking the complement of a given set among the generating operations of the class of positively Borel sets. This economy seems harmless from a classical point of view but intuitionistically it is not. The well-known and beautiful new application of Cantor's diagonal argument by which Borel and Lebesgue prove their Hierarchy Theorem is no longer conclusive.

Each canonical class is shown by them to contain a so-called *universal element*. Forming the diagonal set of the universal element they produce a member of the class the complement of which is not a member of the class.

This complement is not, unfortunately, a positively Borel set.

0.4 One might question the decision to consider positively Borel sets only and to shut out the operation of taking the complement as a generating operation. A reason for doing so, however, is that the set-theoretical operation of taking the complement and the logical operations of negation and implication are not that well understood. G. Gentzen, for one, was of this persuasion and argued from it that the consistency of the formal system of intuitionistic arithmetic is not to be accepted without further proof. Also, a theorem in constructive mathematics is generally judged the more useful and beautiful, the less negations it contains. G.F.C. Griss and D. van Dantzig, under the influence of Brouwer's ideas, distrusted negation and emphasized the importance of so-called *affirmative* mathematics.

0.5 It is possible to prove the Hierarchy Theorem for positively Borel sets? Surprisingly, this question is ignored by P. Martin-Löf in [14] and by E. Bishop and

D. Bridges in [1], although also these authors seem to hold the opinion that, in constructive mathematics, the best Borel sets are positively Borel sets. Brouwer, much earlier studying positively Borel sets of the second level only, sought and found a countable union of closed sets that is not a countable intersection of open sets, and also a countable intersection of open sets that is not a countable union of closed sets, see [2] and [6]. Brouwer's example of a set of the first kind needs some emendation. Also, his proof of its correctness is based upon the Fan Theorem but an elementary argument suffices. (The Fan Theorem is acceptable both from a classical and an intuitionistic point of view, although its intuitionistic proof has been sometimes disputed.) In his proof of the existence of a set of the second kind, Brouwer is making a classically unacceptable assumption: the Continuity Principle. The use of this principle seems unavoidable. Brouwer did not carefully sort out his assumptions, and left this task for others, notably G. Kreisel and S.C. Kleene, see [10] and [12]. Brouwer's treatment of the second level of the Borel Hierarchy is important as it suggests how to formulate the Hierarchy Theorem affirmatively, that is, without using negation, and how to prove it.

0.6 Brouwer's Continuity Principle plays a crucial role in the proof of most of the results of this paper.

Once we agree to accept and use it we enter a new world and discover many facts for which there does not exist a classical counterpart. The principle entails for instance that the union of the two closed sets $[0, 1]$ and $[1, 2]$ is not a countable intersection of open subsets of \mathbb{R} . One may also infer that there are unions of three closed sets different from every union of two closed sets. These observations are the tip of an iceberg. The intuitionistic Borel Hierarchy shows off an exquisite fine structure.

0.7 We prefer to study subsets of Baire space \mathcal{N} rather than subsets of the set \mathbb{R} of real numbers. Results about subsets of \mathcal{N} translate easily into corresponding results about subsets of \mathbb{R} . Given subsets X, Y of \mathcal{N} we say that X *reduces to* Y , notation: $X \preceq Y$, if and only if there exists a continuous function f from \mathcal{N} to \mathcal{N} such that for every α in \mathcal{N} , α belongs to X if and only if $f(\alpha)$ belongs to Y .

The Hierarchy Theorem may be restated in terms of this reducibility relation, as follows: for every positively Borel subset P of \mathcal{N} there exists a positively Borel subset Q of \mathcal{N} such that Q does not reduce to P .

0.8 The contents of this paper are as follows. First, we set out the axioms of intuitionistic analysis. We then discuss the second level of the hierarchy. We go on to consider the notion of reducibility in the class of countable subsets of \mathcal{N} and find the Cantor-Bendixson hierarchy re-emerging in various new ways. In this context we find occasion to introduce the notions Perhaps and Almost. We extend our study into the wider realm of countable unions of closed sets and are amply rewarded. We then

prove the Borel Hierarchy Theorem, and show that the fine structure discovered at the second level is also present at higher levels. We conclude with some remarks on projective sets. Apart from the Introduction, the paper is divided into nine Sections. The titles of the remaining nine Sections are as follows.

1. The axioms and their plausibility.
2. The second step, and our first one.
3. Rediscovering, perhaps, the Cantor-Bendixson hierarchy.
4. Perhaps and Almost.
5. Finite unions of closed sets.
6. Forming limits and finding more hierarchies.
7. The Borel Hierarchy Theorem.
8. The never-ending productivity of disjunction.
9. On (strictly) analytic, co-analytic and projective sets.

0.9 I dedicate this paper to the memory of *Johan J. de Jongh*. He introduced me to Brouwer’s intuitionistic mathematics and asked the question that led to all further ones. As much a philosopher as a mathematician, he hoped to gain insight from precisely and carefully proved mathematical results and also expected that sensible mathematical questions may arise from philosophical reflection. I was made wiser at the seminar on intuitionistic mathematics he conducted in the seventies in Nijmegen. In particular, *Wim Gielen*’s attempts to justify and possibly extend the principles of intuitionistic mathematics were influential and now and then resound in Section 1. I am indebted also to the other participants of this seminar, among them *Harrie de Swart* and *Jo Gielen*.

My later students and, in particular, my Ph.D. students *Tonny Hurkens* and *Frank Waaldijk*, by their enthusiasm and sometimes critical interest, helped me to sustain my belief that intuitionistic mathematics is a worthwhile cause. The notion Perhaps, considered in Sections 3 and 4, grew out of a study of Frank Waaldijk’s notion of “weak stability”, see [25].

0.10 I thank the referee of an earlier version of this paper for spotting several inaccuracies. His comments made me write this second version.

1 The axioms and their plausibility

We explain our point of view and list the assumptions to be used.

1.1 We are contributing to intuitionistic analysis. The logical constants have their constructive meaning and we follow the rules of intuitionistic logic. In particular, a disjunctive statement $A \vee B$ is considered proven only if either A or B is proven and

a proof of an existential statement $\exists x \in V[A(x)]$ has to provide one with a particular element x_0 from the set V and a proof of the corresponding statement $A(x_0)$.

Intuitionistic mathematics distinguishes itself from other varieties of constructive mathematics by its conception of the set of all infinite sequences of natural numbers. This set is not a set in the sense of classical set theory. One does not call it into existence it by bringing together its already existing elements. It could be described tentatively as a kind of frame on which all kinds of projects for constructing infinite sequences of natural numbers may be executed, or, more poetically, as a loom on which all kinds of tapestry may be woven.

The seemingly more simple set of the natural numbers also has to be handled with care. In our view it is a never finished project for producing one by one the natural numbers, $0, 1, 2, \dots$.

Just as the canonical infinite sequence $0, 1, 2, \dots$ *every* infinite sequence α of natural numbers is incomplete all the time and growing step by step as its elements are produced one by one, $\alpha(0), \alpha(1), \alpha(2), \dots$.

The course of the infinite sequence is sometimes dictated by a finitely described algorithm that we keep evaluating. Brouwer's imagination did go further. He contrived, for instance, the following project α : for each n , $\alpha(n) = 0$ if at the moment I want to decide on the value of α in n I have found a proof of Riemann's hypothesis and $\alpha(n) = 1$ if not. The sequence α is thus made to depend on my future experience as a creating mathematical subject. One may be puzzled and even annoyed by this proposal, and remark that unlike an ordinary definition, it does not settle unambiguously the successive values of the sequence. The creating subject still has many decisions to take, for instance, how to count its time. Brouwer, disregarding such problems, and radically pursuing his line of thought envisaged the even more embarrassing possibility of not prescribing anything and allowing the creating subject to choose the successive values of α wholly to its own liking. There is then, as far as we know, no "rule" or "secret plan" governing the development of the sequence, but we, the creating subject, are determined to keep our promise to continue the project and to deliver a next value whenever invited to do so. In such circumstances we never have more information about the infinite sequence than its first finitely many values.

Brouwer did not believe that one may distinguish clearly between algorithmic sequences and non-algorithmic ones. There are sequences that fall between the two stools. Nothing forbids that one starts building a sequence by free choices and then, at some moment, decides to fix its further course by a description in finitely many words.

What is more, we decide to consider infinite sequences from the extensional point of view and deliberately disregard their origin. Every infinite sequence of natural numbers comes into being in many different ways and always, also if it is given by an algorithm, may be imagined to be the result of a free step-by-step construction.

In the following, we endeavour to justify some of the axioms of intuitionistic analysis

from this point of view.

1.2 We present four axioms of countable choice.

We let \mathbb{N} be the set of natural numbers and \mathcal{N} the set of all infinite sequences of natural numbers. We use m, n, \dots as variables over \mathbb{N} , and α, β, \dots as variables over \mathcal{N} . \mathcal{N} is sometimes called *Baire space*. *Cantor space* \mathcal{C} is the set of all α in \mathcal{N} that only assume the values 0,1.

1.2.1 First Axiom of Countable Choice:

For every binary relation R on \mathbb{N} , if for every m there exists n such that mRn , then there exists α such that, for every m , $mR(\alpha(m))$.

We accept this axiom for the following reason.

Suppose we are able to calculate, given any natural number m , a natural number n suitable for m , that is, such that mRn . We then are sure to be able to carry through the project of constructing step by step an infinite sequence α such that, for every m , $\alpha(m)$ is suitable for m . We first choose $\alpha(0)$, then $\alpha(1), \dots$. We do not feel the need to formulate a rule that predicts the choices we will make. Observe that we can not, like non-intuitionistic mathematicians, define α by saying: let $\alpha(m)$ be the least n such that mRn . One may be unable to find the least such n , for instance, if one knows $0R1$ but cannot decide if $0R0$ or not.

1.2.2 Before we introduce a second axiom of countable choice we agree on some notations. \mathbb{N}^* is the set of all finite sequences of natural numbers. We let $\langle \rangle$ be a fixed bijective mapping from \mathbb{N}^* onto \mathbb{N} . Such a function is called a *coding* of the set of finite sequences of natural numbers: $\langle a_0, a_1, \dots, a_{k-1} \rangle$ is the code number of the finite sequence $(a_0, a_1, \dots, a_{k-1})$. We assume that the empty sequence is coded by the number 0 and that for each finite sequence $(a_0, a_1, \dots, a_{k-1})$, for every $i < k$, the code number $\langle a_0, a_1, \dots, a_{k-1} \rangle$ is greater than a_i . We let $*$ denote concatenation: $*$ is a binary function on \mathbb{N} such that, for all m, n , $m * n$ is the code number of the finite sequence obtained by putting the sequence coded by n behind the sequence coded by m .

For all m, n , m is an *initial part* of n if and only if there exists p such that $n = m * p$; and n is an *immediate successor* of m if and only if there exists p such that $n = m * \langle p \rangle$. We also define another function, called J , from $\mathbb{N} \times \mathbb{N}$ to \mathbb{N} : for all m, n : $J(m, n) := \langle m \rangle * n$. It is easy to see that J is a bijective mapping from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N} \setminus \{0\}$.

We let K, L be the inverse functions of J , that is, K and L are functions from $\mathbb{N} \setminus \{0\}$ to \mathbb{N} and for each m , $m \neq 0$: $J(K(m), L(m)) = m$.

J is a non-surjective *pairing function* on \mathbb{N} .

We define, for all α , for all m, n , $\alpha^m(n) := \alpha(J(m, n))$. α^m is called *the m -th subsequence of α* . We also define, for all α , for all m, n , $\alpha^{m,n} := (\alpha^m)^n$.

1.2.3 Second Axiom of Countable Choice:

For every binary relation $R \subseteq \mathbb{N} \times \mathcal{N}$, if for every m there exists α such that $mR\alpha$, then there exists α such that, for every m , $mR(\alpha^m)$.

We accept this axiom for the following reason.

Suppose we are able to calculate, given any natural number m , an infinite sequence α of natural numbers *suitable for* m , that is such that $mR\alpha$. We then form the project of building an infinite sequence α such that for every m , α^m will be suitable for m . We again construct this α step by step. The difficulty of the construction is only slightly greater than in the case of the First Axiom of Countable Choice. We have to start and keep going an infinite number of never finished constructions, one for α^0 , one for α^1, \dots , and so on. At each stage exactly one of these constructions is brought one step further: at stage n one defines $\alpha^{K(n)}(L(n))$.

1.2.4 First Axiom of Dependent Choices:

For every subset X of \mathbb{N} , for every binary relation $R \subseteq X \times X$, if for every m in X there exists n in X such that mRn , then for every m in X there exists α such that $\alpha(0) = m$ and, for every n , $\alpha(n)R\alpha(n+1)$.

We accept this axiom for the following reason.

Suppose we are given at least one element of X , say m . Also assume that for every element p of X we are able to calculate an element n of X *suitable for* p , that is, such that pRn . We then start building a sequence α step by step, first defining $\alpha(0) = m$, then finding $\alpha(1)$ suitable for $\alpha(0)$, then finding $\alpha(2)$ suitable for $\alpha(1)$, and so on. The First Axiom of Dependent Choices will be used in the proof of Lemma 8.5.

1.2.5 Second Axiom of Dependent Choices:

For every subset X of \mathcal{N} , for every binary relation $R \subseteq X \times X$, if for every α in X there exists β in X such that $\alpha R \beta$, then for every α in X there exists β such that $\beta^0 = \alpha$ and, for every n , $\beta^n R \beta^{n+1}$.

We accept this axiom for the same reason as the previous one. We do not think it important that, this time, the objects to be chosen are infinite sequences of natural numbers rather than natural numbers. We state this axiom for completeness' sake, as we will have no occasion to use it in this paper.

1.3 The following axiom is classically false. It makes that intuitionistic analysis is not a subsystem of classical analysis.

1.3.1 We define, given any α and any n , $\bar{\alpha}(n) := \langle \alpha(0), \alpha(1), \dots, \alpha(n-1) \rangle$. If confusion is unlikely to arise, we sometimes write $\bar{\alpha}n$ for $\bar{\alpha}(n)$.

We also define, given any α and any s , s is an initial part of α , or: α passes through s , if and only if, for some n , $\overline{\alpha}n = s$.

1.3.2 Brouwer's Continuity Principle:

For every binary relation $R \subseteq \mathcal{N} \times \mathbb{N}$, if for every α there exists m such that $\alpha R m$, then for every α there exist m, n such that for every β , if $\overline{\alpha}n = \overline{\beta}n$, then $\beta R m$.

We accept this axiom for the following reason.

Suppose we are able to calculate, given any infinite sequence α of natural numbers, a natural number m suitable for α , that is, such that $\alpha R m$. We are attaching the strongest possible meaning both to the “for every α ” and to the “there exists”. Given any infinite sequence whatsoever from the wildly unsurveyable set \mathcal{N} we know how to effectively discover a natural number suitable for it. In particular we can find a suitable number if the sequence is created step by step. A number m , suitable for an α that is given step by step will be found at some moment of time, and at that moment only finitely many values of α , say $\alpha(0), \alpha(1), \dots, \alpha(n-1)$ will be known. The number m will therefore suit every β that has its first n values the same as α . We repeat the remark we made at the end of Section 1.1: every α , even an algorithmically given one, can be thought of as resulting from a free step-by-step-construction. Brouwer's Continuity Principle will be used time and again.

1.3.3 Let X be a subset of \mathcal{N} .

We let the *closure* of X , notation \overline{X} , be the set of all α in \mathcal{N} such that for each n there exists β in X passing through $\overline{\alpha}n$.

X is *weakly-closed* if and only if X coincides with its closure \overline{X} .

X is a *spread* if and only if X is weakly closed and, in addition, we may decide, for every natural number s , if s contains an element of X or not.

Deviating from Brouwer's usage, we also want to call the empty set a spread.

1.3.4 We let **Fun** be the set of all γ such that, for every α , there exists n such that $\gamma(\overline{\alpha}n) \neq 0$. For every γ in **Fun**, every α , we let $\gamma(\alpha)$ be the natural number p such that there exists n such that $\gamma(\overline{\alpha}n) = p+1$ and, for every $m < n$, $\gamma(\overline{\alpha}m) = 0$. In this way, every γ in **Fun** acts as a code for a continuous function from \mathcal{N} to \mathbb{N} .

Observe that, if γ belongs to **Fun** and $\gamma(0) = 0$, then for each n , γ^n belongs to **Fun**. For every γ in **Fun** such that $\gamma(0) = 0$, for every infinite sequence α we define an infinite sequence $\gamma|\alpha$ as follows: for each n , $(\gamma|\alpha)(n) := \gamma^n(\alpha)$. In this way, every γ in **Fun** such that $\gamma(0) = 0$ acts as a code for a continuous function from \mathcal{N} to \mathcal{N} .

1.3.5 Let $X \subseteq \mathcal{N}$ be a non-empty spread.

We intend to define an element r_X of **Fun** with the following two properties:

- (i) $r_X(0) = 0$ and for each α , $r_X|_\alpha$ belongs to X
- (ii) For each α in X , $r_X|_\alpha$ coincides with α .

r_X will be called the *canonical retraction* of \mathcal{N} onto X .

In order to define r_X we first define δ such that $\delta(0) = 0$ and for each s, n , $\delta(s * \langle n \rangle) := \delta(s) * \langle n \rangle$ if $\delta(s) * \langle n \rangle$ contains an element of X , and $\delta(s * \langle n \rangle) := \delta(s) * \langle p \rangle$ where p is the least natural number q such that $\delta(s) * \langle q \rangle$ contains an element of X , if $\delta(s) * \langle n \rangle$ does not contain an element of X . It is easy to see that δ is well-defined and that for each s , if s contains an element of X , then $\delta(s) = s$.

We now may determine r_X in such a way that for every α , every n , $r_X|_\alpha$ passes through $\delta(\overline{\alpha n})$.

Observe that for every α, β, n , if $\overline{\alpha n} = \overline{\beta n}$, then $\overline{r_X|_\alpha n} = \overline{r_X|_\beta n}$.

1.3.6 Theorem: (*Extension of Brouwer's Continuity Principle to spreads*):

Let X be a non-empty spread and R a subset of $X \times \mathbb{N}$.

If for every α in X there exists m such that $\alpha R m$, then for every α in X there exist m, n such that for every β in X , if $\overline{\alpha n} = \overline{\beta n}$, then $\beta R m$.

Proof: Observe that for every α in \mathcal{N} there exists m such that $(r_X|_\alpha) R m$, and apply Brouwer's Continuity Principle. \square

1.3.7 Let X be a spread. We let \mathbf{Fun}_X^0 be the set of all γ such that, for every α in X , there exists n such that $\gamma(\overline{\alpha n}) \neq 0$. For every γ in \mathbf{Fun}_X^0 , every α in X , we let $\gamma(\alpha)$ be the natural number p such that there exists n such that $\gamma(\overline{\alpha n}) = p + 1$, and for every $m < n$, $\gamma(\overline{\alpha m}) = 0$.

We let \mathbf{Fun}_X^1 be the set of all γ such that $\gamma(0) = 0$ and, for each n , γ^n belongs to \mathbf{Fun}_X^0 . For every γ in \mathbf{Fun}_X^1 , for every α in X , we define the element $\gamma|_\alpha$ of \mathcal{N} as follows: for each n , $(\gamma|_\alpha)(n) := \gamma^n(\alpha)$.

If γ belongs to \mathbf{Fun}_X^0 we say that γ is a function from X to \mathbb{N} .

If γ belongs to \mathbf{Fun}_X^1 we say that γ is a function from X to \mathcal{N} .

In particular, an element of \mathbf{Fun} will be called a function from \mathcal{N} to \mathbb{N} , and an element γ of \mathbf{Fun} such that $\gamma(0) = 0$ will be called a function from \mathcal{N} to \mathcal{N} .

Suppose that Z is a subset of \mathcal{N} and γ is a function from X to \mathcal{N} such that for every α in X , $\gamma|_\alpha$ belongs to Z . We then say that γ is a function from X to Z .

1.4 The perception underlying the Continuity Principle may be given a more incisive formulation.

1.4.1 First Axiom of Continuous Choice.

For every binary relation $R \subseteq \mathcal{N} \times \mathbb{N}$,

if for every α there exists m such that $\alpha R m$,

then there exists γ in \mathbf{Fun} such that for every α , $\alpha R(\gamma(\alpha))$.

We accept this axiom for the following reason.

Suppose we are able to find, for every infinite sequence α a natural number m *suitable for* α , that is, such that $\alpha R m$. We allow ourselves to construct the promised γ step by step and consider the (code numbers of the) finite sequences of natural numbers one by one.

Each time we imagine the finite sequence as beginning an infinite sequence that is growing step by step, and we ask ourselves if, as such, it suffices for the determination of a natural number that suits this infinite sequence. If it does, we determine such a number, call it p , and let the value of γ at (the code number of) the finite sequence be $p + 1$, if not, we let that value be 0. We may convince ourselves that for every α , whether it is given by an algorithm or is constructed, more or less freely, step by step, there will exist n such that $\gamma(\overline{\alpha}n) \neq 0$, by reasons similar to the ones that made us accept Brouwer's Continuity Principle.

In this paper, the First Axiom of Continuous Choice is used only in the proof the Finite Borel Hierarchy Theorem, Theorem 7.6.3, but not in the proof of the general Borel Hierarchy Theorem, Theorem 7.7.9.

1.4.2 Second Axiom of Continuous Choice:

For every binary relation $R \subseteq \mathcal{N} \times \mathcal{N}$,
if for every α there exists β such that $\alpha R \beta$,
then there exists γ in **Fun** such that $\gamma(0) = 0$ and $\forall \alpha [\alpha R (\gamma|\alpha)]$.

This axiom implies the two Axioms of Countable Choice and the First Axiom of Continuous Choice. We accept it for the following reason.

Suppose we are able to find, for each infinite sequence α , an infinite sequence β suitable for α , that is, such that $\alpha R \beta$. We construct the promised γ step by step, as follows: we require $\gamma(0) := 0$ and now define, inductively, for each a , the numbers $\gamma^0(a), \gamma^1(a), \dots$ simultaneously.

Our definition will be such that for each n, a , if $\gamma^n(a) \neq 0$, then $n < a$. When considering the code number a of a finite sequence of natural numbers we look for the largest $n \leq a$ with the property that for every $m < n$ there exists an initial part b of a such that $\gamma^m(b) \neq 0$. We call this number n_0 . We imagine the finite sequence coded by a as beginning an infinite sequence α that we are constructing step by step. We managed already to determine the first n_0 values of a sequence β suitable for α and, as we are able to continue the project we started earlier, now ask ourselves if a suffices to determine a next value. If so, we calculate this next value, call it p and define $\gamma^{n_0}(a) := p + 1$, if not, we define $\gamma^{n_0}(a) := 0$. For each $m \neq n_0$, we define $\gamma^m(a) := 0$.

The argument that this procedure guarantees: $\gamma(0) = 0$, γ belongs to **Fun** and $\forall \alpha [\alpha R (\gamma|\alpha)]$ is similar to the argument given for the First Axiom of Continuous Choice and we do not spell it out.

The Second Axiom of Continuous Choice is used in order to prove the collapse of the projective hierarchy, see Theorem 9.16.

1.4.3 Theorem: (Extension of the Axioms of Continuous Choice to spreads).

Let X be a spread.

- (i) For every binary relation $R \subseteq X \times \mathbb{N}$,
if for every α in X there exists m such that $\alpha R m$, then there exists a function γ from X to \mathbb{N} such that for every α in X , $\alpha R (\gamma(\alpha))$.
- (ii) For every binary relation $R \subseteq X \times \mathcal{N}$,
if for every α in X there exists β such that $\alpha R \beta$, then there exists a function from X to \mathcal{N} such that for every α in X , $\alpha R (\gamma|\alpha)$.

Proof: The proof is similar to the proof of Theorem 1.3.6. □

1.5 We need something like countable ordinals and introduce *stumps*. We have taken the word “stump” from [4] but are giving it a slightly different meaning.

For each n , we let \underline{n} be the element of \mathcal{N} with the constant value n .

1.5.1 The set **Stp** of stumps is a subset of Baire space \mathcal{N} and is defined as follows.

- (i) $\underline{1}$ is a stump. We sometimes call $\underline{1}$ the *empty stump*.
- (ii) For all β in \mathcal{N} , if, for each n , β^n is a stump, and $\beta(0) = 0$, then β itself is a stump. We call the stumps β^0, β^1, \dots the *immediate substumps* of the stump β .
- (iii) Clauses (i) and (ii) produce all stumps.

N. Lusin doubted the legitimacy of introducing a set in this way by an inductive definition and Brouwer occasionally expressed similar feelings. We accept the above definition without hesitation, and, as a consequence of (iii), recognize the possibility of giving proofs and constructing functions by transfinite induction on **Stp**.

For every stump β , we define the successor of β , notation: β^+ or $S(\beta)$, by: $(S(\beta))(0) = 0$ and for every n , $(S(\beta))^n = \beta$.

We define a sequence $0^*, 1^*, \dots$ of stumps by induction, as follows. $0^* := \underline{1}$ and for each n , $(n+1)^* := S(n^*)$. Thus we obtain a natural embedding of the set \mathbb{N} into the set **Stp**.

1.5.2 *First Principle of Induction on the set **Stp** of stumps:*

For every subset P of **Stp**, if the empty stump $\underline{1}$ belongs to P , and every non-empty stump β belongs to P as soon as each one of its immediate substumps β^0, β^1, \dots belongs to P , then P coincides with **Stp**.

1.5.3 For every β , for every n , we say that n *belongs to* β if and only if $\beta(n) = 0$. (We are interpreting β as the (inverse) characteristic function of a subset of \mathbb{N}).

Let β be a stump. The set of all finite sequences of natural numbers whose code number belongs to β is more like a “stump” in the sense given to this word by Brouwer. We mention three important properties of this set.

- (i) We may decide, for every finite sequence of natural numbers, if its code number belongs to β or not.
- (ii) Every initial part of a number belonging to β belongs to β .
- (iii) For every γ in \mathcal{N} we may calculate n such that $\bar{\gamma}n$ does not belong to β .

Observe that there is no finite sequence whose code number belongs to $\underline{1}$. Observe that we may decide, for every stump β , if $\beta = \underline{1}$ or not.

1.5.4 *Principle of Stump Induction:*

For every non-empty stump β , for every subset P of \mathbb{N} , if, for every m , m belongs to P as soon as every immediate successor $m * \langle n \rangle$ of m belonging to β belongs to P , then 0 belongs to P .

We leave it to the reader to prove this principle by induction on the set **Stp**.

1.5.5 From now on we use σ, τ, \dots as variables on the set **Stp**. We define binary relations $<, \leq$ on the set **Stp** of stumps as follows: for every stump σ , $\underline{1} \leq \sigma$ and for no stump σ , $\sigma < \underline{1}$. Furthermore, for all stumps σ, τ such that $\tau \neq \underline{1}$, $\tau \leq \sigma$ if and only if, for each n , $\tau^n < \sigma$, and $\sigma < \tau$ if and only if, for some n , $\sigma \leq \tau^n$.

One may prove that the relations $<, \leq$ are transitive and that for all stumps σ, τ , if $\sigma < \tau$, then $\sigma \leq \tau$. Another useful fact is that for all stumps σ, τ, ρ , if $\sigma \leq \tau$ and $\tau < \rho$, then $\sigma < \rho$.

A subset P of **Stp** is called *hereditary* if and only if for every stump σ , σ belongs to P if every $\tau < \sigma$ belongs to P .

1.5.6 *Second Principle of Induction on the set **Stp** of stumps*

Every hereditary subset of **Stp** coincides with **Stp**.

The proof is straightforward.

Observe that this principle does not imply that every inhabited set P of stumps contains an element σ that, for all τ in P , $\sigma \leq \tau$. It is not even true that every inhabited subset of $\{0, 1\}$ has a least element.

Lemma 6.7 enunciates a Principle of Double Induction on the set **Stp** of stumps. Theorem 6.5 enunciates a Principle of Induction on the **Stp**^{*} of finite sequences of stumps.

1.6 We consider the assumption that underlies the famous Bar Theorem. It will play a role in this paper when we come to discuss the notion Almost in Section 4 and also figures in various results about (strictly) analytic and co-analytic sets in Section 9.

1.6.1 A subset P of \mathbb{N} will be called a *bar* if and only if for each α there exists n such that $\bar{\alpha}n$ belongs to P .

A subset P of \mathbb{N} is called *monotone* if and only if for every m , if m belongs to P , then every immediate successor $m * \langle n \rangle$ of m belongs to P .

1.6.2 *Brouwer's Thesis:*

For every subset P of \mathbb{N} ,
if P is a bar then there exists a stump β such that the set of all elements
of P belonging to β is a bar.

Brouwer thought that his Thesis could be seen to be true by reflection on the possible structure of a (canonical) proof of the fact “for every α there exists n such that $P(\bar{\alpha}n)$ ”. We refrain from discussing his argument.

We warn the reader that our formulation of Brouwer's Thesis does not literally occur in Brouwer's writings.

1.6.3 Principle of Induction on Monotone Bars:

Let P be a monotone bar. Let Q be a subset of \mathbb{N} such that $P \subseteq Q$ and for every m , if every immediate successor $m * \langle n \rangle$ of m belongs to P , then m belongs to Q . Then 0 belongs to Q .

One easily proves this from Brouwer's Thesis and the Principle of Induction on Stumps.

1.6.4 Principle of Double Induction on Monotone Bars:

Let P be a subset of \mathbb{N} such that for every α, β there exists n such that $\langle \bar{\alpha}n, \bar{\beta}n \rangle$ belongs to P , and for every s, t , if $\langle s, t \rangle$ belongs to P then, for every m , $\langle s * \langle m \rangle, t \rangle$ and $\langle s, t * \langle m \rangle \rangle$ belong to P . Let Q be a subset of \mathbb{N} such that $P \subseteq Q$ and for every s, t , if for every m both $\langle s * \langle m \rangle, t \rangle$ and $\langle s, t * \langle m \rangle \rangle$ belong to Q , then $\langle s, t \rangle$ belongs to Q . Then $\langle 0, 0 \rangle$ belongs to Q .

The proof is left to the reader.

1.6.5 A famous consequence of Brouwer's Thesis is the *Fan Theorem*.

A *fan* or *finitary spread* is a subset F of Baire space \mathcal{N} such that there exists β with the following two properties:

- (i) for every α , α belongs to F if and only if, for each n , $\beta(\bar{\alpha}n) = 0$, and
- (ii) for each n such that $\beta(n) = 0$ there exists m such that for all k , if $\beta(n * \langle k \rangle) = 0$, then $k < m$.

Let X be a subset of \mathcal{N} and let P be a subset of \mathbb{N} . P is called a *bar in X* if for every α in X there exists n such that $\bar{\alpha}n$ belongs to P .

1.6.6 *Fan Theorem*:

Let $F \subseteq \mathcal{N}$ be a fan.
For every subset P of \mathbb{N} ,
if P is a bar in F , then some finite subset of P is a bar in F .

Brouwer used the Fan Theorem for proving that every real function defined on $[0, 1]$ is uniformly continuous on $[0, 1]$, see [3].

1.7 We show how one may introduce real numbers into intuitionistic analysis.

1.7.1 Let ρ be an enumeration of the set \mathbb{Q} of rational numbers.

Let α belong to \mathcal{N} .

α is called a *real number* if and only if, for each n , $\rho(\alpha(2n)) < \rho(\alpha(2n+2)) < \rho(\alpha(2n+3)) < \rho(\alpha(2n+1))$, and, for every q, r in \mathbb{Q} , if $q < r$, then there exists n such that either $\rho(\alpha(2n+1)) < r$ or $q < \rho(\alpha(2n))$.

α is called a *canonical real number* if, in addition, for each n , if $\rho(K(n+1)) < \rho(L(n+1))$, then either $\rho(\alpha(2n+1)) < \rho(L(n+1))$ or $\rho(K(n+1)) < \rho(\alpha(2n))$. We denote the set of real numbers by \mathbb{R} and the set of canonical real numbers by **Crn**.

1.7.2 Let α, β be real numbers. α is *really-smaller than* β , notation $\alpha <^* \beta$, if and only if there exists n such that $\rho(\alpha(2n+1)) < \rho(\beta(2n))$. α is *really-not-greater than* β , notation $\alpha \leq^* \beta$ if and only if, for each n , $\rho(\alpha(2n)) < \rho(\beta(2n+1))$. α is *really-apart from* β , notation $\alpha \#^* \beta$, if and only if either $\alpha <^* \beta$ or $\alpha <^* \beta$. α *really-coincides* with β , notation $\alpha =^* \beta$ if and only if the assumption $\alpha \#^* \beta$ leads to a contradiction.

We let *the open real interval* (α, β) be the set of all real numbers γ such that $\alpha <^* \gamma <^* \beta$. We let *the closed real interval* $[\alpha, \beta]$ be the set of all real numbers γ such that $\alpha \leq^* \gamma \leq^* \beta$.

Let A, B be subsets of \mathbb{R} .

We say that A *really coincides* with B if and only if every element of A really coincides with an element of B and conversely, every element of B really coincides with an element of A .

1.7.3 Observe that \mathbb{R} really-coincides with **Crn** and that **Crn**, viewed as a subset of \mathcal{N} , is a spread. This fact lies at the basis of the famous result that every real function is continuous, see [3] and [22].

2 The second step, and our first one

We introduce sets from the second level of the Borel Hierarchy and prove the corresponding case of the Hierarchy Theorem.

2.1 Let X be a subset of Baire space \mathcal{N} .

X is *basic open* if and only if either X is empty or there exists s in \mathbb{N} such that X consists of all α passing through s .

X is *open* or Σ_1^0 if and only if there exists a sequence X_0, X_1, \dots of basic open sets such that $X = \bigcup_{n \in \mathbb{N}} X_n$.

X is *closed* or Π_1^0 if and only if there exists an open set Y such that X consists of all α that do not belong to Y .

Every closed subset of \mathcal{N} is weakly closed in the sense of Section 1.3.3, but not conversely. In general, a closed subset of \mathcal{N} is not a spread in the sense of Section 1.3.3, but every spread is a closed subset of \mathcal{N} .

X is Σ_2^0 if and only if there exists a sequence X_0, X_1, \dots of closed sets such that $X = \bigcup_{n \in \mathbb{N}} X_n$.

X is Π_2^0 if and only if there exists a sequence X_0, X_1, \dots of open sets such that $X = \bigcap_{n \in \mathbb{N}} X_n$.

2.2 Let X be a subset of the set \mathbb{R} of real numbers.

X is (*really*) *basic open* if and only if either X is empty or there exists rational numbers p, q such that X is the *open interval* (p, q) , that is, X consists of all α in \mathbb{R} for which there exists n such that $p < \rho(\alpha(2n)) < \rho(\alpha(2n+1)) < q$.

The notions of a (*really*) Σ_1^0 , (*really*) Π_1^0 , (*really*) Σ_2^0 , (*really*) Π_2^0 subset of \mathbb{R} are defined as the corresponding notions for subsets of \mathcal{N} in Section 2.1.

2.3 We let **Rat** be the set of all real numbers α for which there exists a rational number q such that, for every n , $\rho(\alpha(2n)) < q < \rho(\alpha(2n+1))$.

We let **PosIrr** be the set of real numbers α such that for every rational number q there exists n such that either $q < \rho(\alpha(2n))$ or $\rho(\alpha(2n+1)) < q$.

Rat is the set of all real numbers coinciding with a rational, and **PosIrr** is the set of all *positively irrational* numbers.

Observe that every element of **Rat** is really-apart from every element of **PosIrr**.

2.4 Theorem: For every sequence X_0, X_1, \dots of open subsets of \mathbb{R} , if **Rat** is a subset of $\bigcap_{n \in \mathbb{N}} X_n$, then some element of **PosIrr** belongs to $\bigcap_{n \in \mathbb{N}} X_n$.

Proof: Suppose that **Rat** is a subset of $\bigcap_{n \in \mathbb{N}} X_n$.

We construct a real number α such that, for each n , the open interval $(\rho(\alpha(2n)), \rho(\alpha(2n+1)))$ is a subset of X_n , and either $\rho(n) < \rho(\alpha(2n))$, or $\rho(\alpha(2n+1)) < \rho(n)$. \square

2.5 Theorem: For every sequence X_0, X_1, \dots of closed subsets of \mathbb{R} , if **PosIrr** is a subset of $\bigcup_{n \in \mathbb{N}} X_n$, then some element of **Rat** belongs to $\bigcup_{n \in \mathbb{N}} X_n$.

Proof: We let **Pir** be the set of all real numbers α such that for each n , either $\rho(n) < \rho(\alpha(2n))$ or $\rho(\alpha(2n+1)) < \rho(n)$. Observe that **Pir** is a subset of **PosIrr** and that **Pir**, viewed as a subset of \mathcal{N} , is a spread. In addition, **PosIrr** really-coincides with **Pir**. Suppose that **PosIrr** is a subset of $\bigcup_{n \in \mathbb{N}} X_n$, and let α_0 be some element of **Pir**. Applying the Continuity Principle, we find m, n such that for all α in **Pir**, if $\bar{\alpha}(2m) = \bar{\alpha}_0(2m)$ then α belongs to X_n . We conclude that every positively irrational number in the open interval $(\rho(\alpha(2m-2)), \rho(\alpha(2m-1)))$ belongs to the closed set X_n , therefore also every rational number in this interval belongs to X_n . \square

2.6 Corollary: **Rat** is Σ_2^0 but not Π_2^0 , and **PosIrr** is Π_2^0 but not Σ_2^0 .

Proof: **Rat** is a countable union of singletons. Every singleton is the set of all elements of \mathbb{R} really-coinciding with a given real number and is a closed subset of \mathbb{R} .

Therefore **Rat** is Σ_2^0 , whereas, according to Theorem 2.4, **Rat** is not Π_2^0 .

PosIrr is the intersection of countably many sets of the form

$\{\alpha \mid \alpha \in \mathbb{R} \text{ for some } n, \text{ either } q < \alpha(2n) \text{ or } \alpha(2n+1) < q\}$, where q is a given rational number, and every such set is an open subset of \mathbb{R} . Therefore **PosIrr** is Π_2^0 , whereas, according to Theorem 2.5, **PosIrr** is not Σ_2^0 . \square

2.7 Rat is a not too difficult example of a set that belongs to the class Σ_2^0 but not to the class Π_2^0 .

Brouwer describes a more complicated example in [6]. He considers a set $\bigcup_{n \in \mathbb{N}} K_n$,

where, for each n , K_n is the set of all real numbers α for which there exists a sequence $\beta(0), \beta(1), \dots$ such that for each i , $\beta(i)$ belongs to $\{0, 1, 2\}$, and for each m , the set $\{i \mid i < m \mid \beta(i) = 1\}$ has at most n members, and α really-coincides with $\sum_{i \in \mathbb{N}} \beta(i) \cdot 3^{-i-1}$,

(so α belongs to the closed interval $[0, 1]$ and has a ternary expansion in which the number 1 occurs at most n times). Brouwer assumes that every set K_n is a closed subset of \mathbb{R} , but this is only true if $n = 0$. This oversight was observed and pointed out to me by the referee of an earlier version of this paper. We may correct it by replacing every set K_n by its closure \overline{K}_n , that is the set of all real numbers α that may be obtained as the limit of a convergent sequence x_0, x_1, \dots where each x_i is a rational number of the form $\sum_{i=0}^m b(i) \cdot 3^{-i-1}$ and each $b(i)$ belongs to $\{0, 1, 2\}$ and the set $\{i \mid i < m \mid b(i) = 1\}$ has at most n members. In general, an element of \overline{K}_n does not have a ternary expansion and does not belong to K_n .

In his proof that the set $\bigcup_{n \in \mathbb{N}} \overline{K}_n$ is not Π_2^0 , Brouwer unnecessarily applies the Fan Theorem.

Brouwer proposes the set K_∞ as an example of a set that is Π_2^0 and not Σ_2^0 . K_∞ is the set of all numbers α in the closed interval $[0, 1]$ with a ternary expansion in which the number 1 occurs infinitely many times.

In proving this example correct, Brouwer uses the Continuity Principle, like we did in the proof of Theorem 2.5. Observe that K_∞ , like \mathbb{R} itself, really-coincides with a spread.

2.8 We return to Baire space \mathcal{N} .

For all α, β in \mathcal{N} , we define: α is apart from β , or: α lies apart from β , notation: $\alpha \# \beta$, if and only if there exists n such that $\alpha(n) \neq \beta(n)$. This constructive inequality relation is *co-transitive*, that is, for all α, β, γ , if $\alpha \# \beta$, then either $\alpha \# \gamma$ or $\gamma \# \beta$.

Let γ belong to \mathcal{N} . Recall from Section 1.3.7 that γ is a *function from \mathcal{N} to \mathcal{N}* if and only if γ belongs to **Fun** and $\gamma(0) = 0$.

Let X, Y be subsets of \mathcal{N} and let γ be a function from \mathcal{N} to \mathcal{N} .

γ maps X into Y if and only if for every α in X , $\gamma|\alpha$ belongs to Y .

γ reduces X to Y if and only if γ maps X and only X into Y , that is for every α , α

belongs to X if and only if $\gamma|\alpha$ belongs to Y .

X is *reducible* to Y , or X *reduces* to Y , notation $X \preceq Y$, if and only if some γ in **Fun** reduces X to Y .

If γ reduces X to Y , then γ can be considered as an effective method to translate every question: “does α belong to X ?” into a question: “does β belong to Y ?” This notion is called “Wadge-reducibility” in classical descriptive set theory. Its analogue in recursion theory is called “many-one-reducibility” or “ m -reducibility”. We use the unadorned expression “reducible” as no other notion of reducibility figures in this paper.

2.9 We define subsets A_2 and E_2 of \mathcal{N} as follows.

A_2 is the set of all α such that for every m there exists n such that $\alpha^m(n) \neq 0$.

E_2 is the set of all α such that for some m , for every n , $\alpha^m(n) = 0$. Observe that every element of A_2 is apart from every element of E_2 .

2.10 Theorem: Let X be a subset of \mathcal{N} .

- (i) X belongs to Π_2^0 if and only if X reduces to A_2 .
- (ii) X belongs to Σ_2^0 if and only if X reduces to E_2 .

Proof: We leave the straightforward proof to the reader. □

2.11 Theorem: Every function from \mathcal{N} to \mathcal{N} that maps E_2 into A_2 also maps some element of A_2 into A_2 .

Proof: Let γ be a function from \mathcal{N} to \mathcal{N} that maps E_2 into A_2 . We now construct α such that both α and $\gamma|\alpha$ belong to A_2 . First define $\alpha_0 := \underline{0}$. Then α_0 belongs to E_2 , therefore $\gamma|\alpha_0$ belongs to A_2 . Calculate n_0 such that $(\gamma|\alpha)^0(n_0) \neq 0$. Also calculate m_0 such that for every β , if $\bar{\beta}m_0 = \bar{\alpha}_0m_0$, then $(\gamma|\beta)^0(n_0) = (\gamma|\alpha_0)^0(n_0)$. Now define α_1 such that $\alpha_1(\langle 0, m_0 \rangle) = 1$ and $\bar{\alpha}_1m_0 = \bar{\alpha}_0m_0$ and $(\alpha_1)^1 = \underline{0}$. Then α_1 belongs to E_2 , therefore $\gamma|\alpha_1$ belongs to A_2 . Calculate n_1 such that $(\gamma|\alpha_1)^1(n_1) \neq 0$. Also calculate m_1 such that $m_1 > \langle 0, m_0 \rangle$ and for every β , if $\bar{\beta}m = \bar{\alpha}_1m_1$, then $(\gamma|\beta)^1(n_1) = (\gamma|\alpha_1)^1(n_1)$.

Now define α_2 such that $\bar{\alpha}_2m_1 = \bar{\alpha}_1m_1$ and $\alpha_2(\langle 1, m_1 \rangle) = 1$ and $(\alpha_2)^2 = \underline{0}$. Then α_2 belongs to E_2 , therefore $\gamma|\alpha_2$ belongs to A_2 . Continuing in this way, we find two sequences n_0, n_1, \dots and m_0, m_1, \dots of natural numbers and a sequence $\alpha_0, \alpha_1, \dots$ of elements of \mathcal{N} such that $m_0 < m_1 < \dots$ and for each k , $\bar{\alpha}_{k+1}m_k = \bar{\alpha}_km_k$ and for each k , for each β , if $\bar{\beta}m_k = \bar{\alpha}_km_k$, then $(\gamma|\beta)^k(n_k) \neq 0$ and if $\bar{\beta}m_{k+1} = \bar{\alpha}_{k+1}m_{k+1}$, then $\beta^k(m_k) = 1$. Consider the sequence α such that for each k , $\bar{\alpha}m_k = \bar{\alpha}_km_k$ and observe: both α and $\gamma|\alpha$ belong to A_2 . □

2.12 Theorem: There exists a function f from \mathcal{N} to \mathcal{N} with the following properties:

- (i) For every α , $f|\alpha$ belongs to A_2 .
- (ii) For every β in A_2 there exists α such that $f|\alpha = \beta$.
- (iii) For every n , α , there exists m such that for every β , if $\bar{\beta}m = (\overline{f|\alpha})m$ and β belongs to A_2 , then there exists γ such that $\bar{\gamma}n = \bar{\alpha}n$ and $\beta = f|\gamma$.

Proof: Observe that, for every β , β belongs to A_2 if and only if for each m there exists n such that $\beta^m(n) \neq 0$ if and only if there exists δ such that for each m , $\beta^m(\delta(m)) \neq 0$.

We now define a function f from \mathcal{N} to \mathcal{N} such that for every α , $(f|\alpha)(0) = \alpha^1(0)$ and for every m , $(f|\alpha)^m(\alpha^0(m)) = \text{Max}(1, \alpha^{1,m}(\alpha^0(m)))$ and for every m, k , if $k \neq \alpha^0(m)$, then $(f|\alpha)^m(k) = \alpha^{1,m}(k)$. One verifies easily that f has the promised properties. \square

2.13 Theorem: Every function from \mathcal{N} to \mathcal{N} that maps A_2 into E_2 , also maps some element of E_2 into E_2 .

Proof: Let γ be a function from \mathcal{N} to \mathcal{N} that maps A_2 into E_2 . Let f be a function from \mathcal{N} to \mathcal{N} with the properties mentioned in Theorem 2.12. Observe that for each α we may calculate n such that $(\gamma|(f|\alpha))^n = \underline{0}$. Applying the Continuity Principle we find m, n such that for every α , if $\bar{\alpha}m = \underline{0}m$ then $(\gamma|(f|\alpha))^n = \underline{0}$. We calculate p such that for every β , if $\bar{\beta}p = (\overline{f|\underline{0}})p$ and β belongs to A_2 , then there exists α passing through $\underline{0}m$ such that $\beta = f|\alpha$. We now define δ in such a way that $\bar{\delta}p = (\overline{f|\underline{0}})p$ and $\delta^p = \underline{0}$. Observe that also $(\gamma|\delta)^n = \underline{0}$, as for each q there exists α such that $\bar{\alpha}q = \bar{\delta}q$ and $(f|\alpha)^n = \underline{0}$.

Therefore both δ and $\gamma|\delta$ belong to E_2 . \square

2.14 Corollary: The set E_2 belongs to Σ_2^0 but not to Π_2^0 .
The set A_2 belongs to Π_2^0 but not to Σ_2^0 .

Proof: Obvious. \square

3 Rediscovering, perhaps, the Cantor-Bendixson hierarchy

We study a special collection of enumerable sets and show how, in various ways, they form a hierarchy. We also introduce a binary operation, called Perhaps, on the class of subsets of \mathcal{N} .

3.1 For every s in \mathbb{N} , every α , we let $s * \alpha$ be the element of \mathcal{N} that we obtain by putting the infinite sequence α behind the finite sequence (coded by) s .

For every s in \mathbb{N} , every subset X of \mathcal{N} we let $s * X$ be the set of all infinite sequences $s * \alpha$, where α belongs to X . We now define, for every stump σ , a subset CB_σ of \mathcal{N} by means of the following inductive definition:

- (i) $CB_{\perp} := \emptyset$
- (ii) For every non-empty stump σ , $CB_{\sigma} := \{\underline{0}\} \cup \bigcup_{n \in \mathbb{N}} \overline{0}n * \langle 1 \rangle * CB_{\sigma^n}$.

The letters CB have been chosen in honour of Cantor and Bendixson.

3.2 Let X be a subset of \mathcal{N} . Recall that the *closure* of X , notation \overline{X} , is the set of all α in \mathcal{N} such that for every n there exists an element of X passing through $\overline{\alpha}n$. If we may decide, for every s , if s contains an element of X or not, then \overline{X} is a closed subset of \mathcal{N} and a spread.

3.3 Theorem:

- (i) For every stump σ , $\overline{CB_{\sigma}}$ is a spread.
- (ii) For every non-empty stump σ , if $\overline{CB_{\sigma}}$ coincides with CB_{σ} , then CB_{σ} is a finite subset of \mathcal{N} .

Proof: We leave the proof of (i) to the reader.

As to (ii), assume that $\overline{CB_{\sigma}}$ coincides with CB_{σ} . Observe that now, for every α in $\overline{CB_{\sigma}}$, *either* $\alpha = \underline{0}$, *or*, for some n , $\alpha(n) = 1$. $\overline{CB_{\sigma}}$ is a spread containing $\underline{0}$. Applying the Continuity Principle we find n such *either* every α in $\overline{CB_{\sigma}}$ passing through $\overline{0}n$ coincides with $\underline{0}$, *or* every α in $\overline{CB_{\sigma}}$ passing through $\overline{0}n$ is apart from $\underline{0}$. Both alternatives are absurd as both $\underline{0}$ and $\overline{0}n * \langle 1 \rangle * \underline{0}$ belong to CB_{σ} . \square

3.4 Let X be a subset of \mathcal{N} .

X will be called *enumerable* if and only if either X is empty or there exists an *enumeration* of X , that is, an element α of \mathcal{N} such that X coincides with the set $\{\alpha^0, \alpha^1, \dots\}$. Every enumerable subset of \mathcal{N} belongs to the class Σ_2^0 .

Given any subset X of \mathcal{N} , we let the *complement* of X , notation X^{\neg} , be the set of all α such that α does not belong to X , that is, the assumption that α does belong to X leads to a contradiction.

Every subset X of \mathcal{N} forms part of its double complement $X^{\neg\neg}$ but the converse is not generally true. On the other hand, for every subset X of \mathcal{N} , $X^{\neg\neg\neg}$ coincides with X^{\neg} .

3.5 Theorem:

- (i) For every stump σ , CB_{σ} is an enumerable subset of \mathcal{N} .
- (ii) For every σ , the set $\overline{CB_{\sigma}}$ coincides with the set $(CB_{\sigma})^{\neg\neg}$.

Proof: We leave the proof of (i) to the reader.

As to (ii), first remark that for every σ , $(CB_{\sigma})^{\neg\neg}$ is a subset of $\overline{CB_{\sigma}}$, as for every α in $(CB_{\sigma})^{\neg\neg}$, for every n , $\overline{\alpha}n$ contains a member of CB_{σ} .

In order to prove that for every σ , the set $\overline{CB_{\sigma}}$ forms part of the set $(CB_{\sigma})^{\neg\neg}$ we use induction on the set of stumps.

Observe that the set $CB_{\underline{1}}$ coincides with its closure $\overline{CB_{\underline{1}}}$ and with its double complement $(CB_{\underline{1}})^{\neg\neg}$.

Now assume that σ is a non-empty stump and that, for each n , the set $\overline{CB_{\sigma^n}}$ forms part of the set $(CB_{\sigma^n})^{\neg\neg}$. Let α belong to $\overline{CB_{\sigma}}$. We distinguish two cases. *First case:* $\alpha = \underline{0}$. Then α belongs to CB_{σ} . *Second case:* $\alpha \neq \underline{0}$. Calculate n, β such that $\alpha = \underline{0}n * \langle 1 \rangle * \beta$. Observe that β belongs to $\overline{CB_{\sigma^n}}$ and therefore also to $(CB_{\sigma^n})^{\neg\neg}$. Therefore α belongs to $(CB_{\sigma})^{\neg\neg}$.

We thus see that if either $\alpha = \underline{0}$ or $\alpha \neq \underline{0}$, then α belongs to $(CB_{\sigma})^{\neg\neg}$. We recall that $\neg\neg(\alpha = \underline{0} \vee \alpha \neq \underline{0})$ and conclude that α belongs to $(CB_{\sigma})^{\neg\neg\neg\neg} = (CB_{\sigma})^{\neg\neg}$. Therefore $\overline{CB_{\sigma}}$ forms part of $(CB_{\sigma})^{\neg\neg}$. \square

3.6 Observe that from a classical point of view, the sets CB_{σ} are closed subsets of \mathcal{N} .

We have just seen that, except for $CB_{\underline{1}}$, the sets CB_{σ} are not closed. Being countable, they do belong to the class Σ_2^0 .

Let X be a subset of \mathcal{N} and P a subset of \mathbb{N} . Recall that P is called a *bar in X* if and only if every element of X has an initial part in P .

3.7 Theorem:

- (i) For every stump σ , for every subset P of \mathbb{N} , if P is a bar in the countable set CB_{σ} , then some finite subset of P is a bar in CB_{σ} .
- (ii) For every stump σ , for every open subset Y of \mathcal{N} , if the set CB_{σ} forms part of Y , then its closure $\overline{CB_{\sigma}}$ forms part of Y .
- (iii) For every non-empty stump σ , if the set CB_{σ} belongs to the class Π_2^0 , then CB_{σ} is a finite subset of \mathcal{N} .

Proof: (i) We use induction on the set of stumps. The empty set is a bar in $CB_{\underline{1}}$. Assume that σ is a non-empty stump and the statement holds true for every set CB_{σ^n} . Also assume that P is a bar in CB_{σ} . Calculate m such that $\underline{0}m$ belongs to P . Using the induction hypothesis, find finite subsets P_0, \dots, P_{m-1} of P such that for each $j < m$, P_j is a bar in the set of all α in CB_{σ} passing through $\underline{0}j * \langle 1 \rangle$. The set $Q := \{\underline{0}m\} \cup \bigcup_{j < m} P_j$ will satisfy our purposes.

(ii) Let Y be an open subset of \mathcal{N} . Let P be the set of all s in \mathbb{N} such that every α passing through s belongs to Y . Suppose that σ is a stump and CB_{σ} forms part of Y . Then P is a bar in CB_{σ} . Determine a finite subset Q of P such that Q is a bar in CB_{σ} and observe that Q is also a bar in $\overline{CB_{\sigma}}$. Therefore $\overline{CB_{\sigma}}$ forms part of Y .

(iii) Assume that σ is a non-empty stump and CB_{σ} belongs to Π_2^0 . Determine a sequence G_0, G_1, \dots of open subsets of \mathcal{N} such that $CB_{\sigma} = \bigcap_{n \in \mathbb{N}} G_n$. According to

(ii), $\overline{CB_{\sigma}} \subseteq \bigcap_{n \in \mathbb{N}} G_n$, therefore $CB_{\sigma} = \overline{CB_{\sigma}}$ and, by Theorem 3.3, CB_{σ} is a finite subset of \mathcal{N} . \square

3.8 Let X, Y be spreads and let γ be a function from X to Y .

γ *embeds* X into Y , or: γ is an *embedding of X into Y* if and only if for all α, β in X , if α is apart from β , then $\gamma|\alpha$ is apart from $\gamma|\beta$. X *embeds into* Y if and only if some γ embeds X into Y .

Let β belong to \mathcal{N} . β is *repetitive* if and only if for each m there exists n such that $n > m$ and $\beta^n = \beta^m$.

We introduce the set **Hrs** of *hereditarily repetitive stumps*.

Hrs is a subset of **Stp** and is given by the following inductive definition:

- (i) $\underline{1}$ is a hereditarily repetitive stump.
- (ii) For all β , if, for each n , β^n is a hereditarily repetitive stump, and β is repetitive, and $\beta(0) = 0$, then β itself is a hereditarily repetitive stump.
- (iii) Clauses (i) and (ii) produce all hereditarily repetitive stumps.

3.9 Theorem:

- (i) For all hereditarily repetitive stumps σ, τ , if $\overline{CB_\sigma}$ embeds into $\overline{CB_\tau}$, then $\sigma \leq \tau$.
- (ii) For all hereditarily repetitive stumps σ, τ , if $\sigma < \tau$, then CB_τ does not reduce to CB_σ .
- (iii) For all hereditarily repetitive stumps σ, τ , if $\sigma \leq \tau$ then both $\overline{CB_\sigma}$ embeds into $\overline{CB_\tau}$ and CB_σ reduces to CB_τ .

Proof: (i) We use induction on the set of hereditarily repetitive stumps. For each hereditarily repetitive stump σ we define the proposition $P(\sigma)$ as follows:

$$\begin{aligned} P(\sigma) &:= \sigma \text{ has the property } P \\ &:= \text{for every hereditarily repetitive stump } \tau, \\ &\quad \text{if } \overline{CB_\sigma} \text{ embeds into } \overline{CB_\tau}, \text{ then } \sigma \leq \tau. \end{aligned}$$

Observe that $P(\underline{1})$ is obviously true. Now assume that σ is a non-empty hereditarily repetitive stump and for each n , $P(\sigma^n)$.

Assume that τ is a hereditarily repetitive stump and γ embeds $\overline{CB_\sigma}$ into $\overline{CB_\tau}$. Let m belong to \mathbb{N} . Calculate p such that $p > m$ and $\sigma^p = \sigma^m$.

Observe that either $\gamma|(\underline{0}m * \langle 1 \rangle * \underline{0})$ is apart from $\underline{0}$ or $\gamma|(\underline{0}p * \langle 1 \rangle * \underline{0})$ is apart from $\underline{0}$. Let us assume the latter and calculate q such that, $\gamma|(\underline{0}p * \langle 1 \rangle * \underline{0})$ equals $\underline{0}q * \langle 1 \rangle * \alpha$, for some α . Calculate r such that for every α in $\overline{CB_\sigma}$, if α passes through $\underline{0}p * \langle 1 \rangle * \underline{0}r$, then $\gamma|\alpha$ passes through $\underline{0}q * \langle 1 \rangle$. Let δ be a function from CB_{σ^p} to \mathcal{N} such that, for every α in CB_{σ^p} , γ maps the sequence $\underline{0}p * \langle 1 \rangle * \underline{0}r * \alpha$ onto $\underline{0}q * \langle 1 \rangle * \delta|\alpha$. Observe that δ embeds CB_{σ^p} into CB_{τ^q} , therefore $\sigma^m = \sigma^p \leq \tau^q$.

The case that $\gamma|(\underline{0}m * \langle 1 \rangle * \underline{0})$ is apart from $\underline{0}$ is treated similarly.

We conclude $\forall m \exists q[\sigma^m \leq \tau^q]$, therefore $\sigma \leq \tau$.

Therefore every hereditarily repetitive stump has the property P .

(ii) We again use induction on the set of hereditarily repetitive stumps. For each hereditarily repetitive stump σ we define the proposition $Q(\sigma)$ as follows:

$$\begin{aligned} Q(\sigma) &:= \sigma \text{ has the property } Q \\ &:= \text{For every hereditarily repetitive stump } \sigma, \\ &\quad \text{if } \sigma < \tau, \text{ then } CB_\tau \text{ does not reduce to } CB_\sigma. \end{aligned}$$

$Q(\underline{1})$ is obvious. Assume that σ is a non-empty hereditarily repetitive stump and, for each n , $Q(\sigma^n)$. Also assume that τ is a hereditarily repetitive stump and $\sigma < \tau$ and γ is a function from \mathcal{N} to \mathcal{N} reducing CB_τ to CB_σ . Calculate q such that $\sigma \leq \tau^q$. We claim that γ maps $\underline{0}q * \langle 1 \rangle * \underline{0}$ onto $\underline{0}$. For suppose $\gamma|(\underline{0}q * \langle 1 \rangle * \underline{0})$ is apart from $\underline{0}$ and determine m such that $\gamma|(\underline{0}q * \langle 1 \rangle * \underline{0})$ equals $\underline{0}m * \langle 1 \rangle * \alpha$, for some α . Calculate t such that for every β , if β passes through $\underline{0}q * \langle 1 \rangle * \underline{0}t$ then $\gamma|\beta$ passes through $\underline{0}m * \langle 1 \rangle$. Construct a function δ from \mathcal{N} to \mathcal{N} such that, for every α , $\gamma|(\underline{0}q * \langle 1 \rangle * \underline{0}t * \alpha)$ equals $\underline{0}m * \langle 1 \rangle * (\delta|\alpha)$ and observe that δ reduces the set of all α such that $\underline{0}t * \alpha$ belongs to CB_{τ^q} to CB_{σ^m} . The set CB_{τ^q} is easily seen to reduce to the set of all α such that

$\bar{0}t * \alpha$ belongs to CB_{τ^q} and therefore CB_{τ^q} itself reduces to CB_{σ^m} . We now have a contradiction, as $\sigma^m < \sigma \leq \tau^q$, therefore $\sigma^m < \tau^q$, and, by the induction hypothesis, CB_{τ^q} does not reduce to CB_{σ^m} .

Now τ is hereditarily repetitive, so there exists a strictly increasing sequence $q = q_0 < q_1 < \dots$ such that for each n , $\tau^{q_n} = \tau^q$, and therefore γ maps $\bar{0}q_n * \langle 1 \rangle * \bar{0}$ onto $\bar{0}$. Consider the closure X of the set $\{\bar{0}q_n * \langle 1 \rangle * \bar{0} | n \in \mathbb{N}\}$. Observe that X is a spread containing $\bar{0}$ and that γ maps every element of X onto $\bar{0}$, and thus into CB_σ , therefore, as γ reduces CB_τ to CB_σ , X is a subset of CB_τ , and every α in X either coincides with $\bar{0}$ or is apart from $\bar{0}$. Applying the Continuity Principle we find m such that *either* every α in X passing through $\bar{0}m$ coincides with $\bar{0}$ *or* every α in X passing through $\bar{0}m$ is apart from $\bar{0}$, an obvious contradiction.

We conclude that CB_τ does not reduce to CB_σ .

Therefore every hereditarily repetitive stump has the property Q .

(iii) The proof is left to the reader. \square

3.10 Some of the results we obtained thus far contrast starkly with some theorems in classical descriptive set theory: there, by a result of W. Wadge, see [11], p. 169, every set X that belongs to Σ_2^0 but not to Π_2^0 is Σ_2^0 -complete, that is, every set belonging to Σ_2^0 reduces to X . As a consequence, all sets that belong to Σ_2^0 but not to Π_2^0 are of the same reducibility degree. Here, we find large hierarchies formed by sets from $\Sigma_2^0 \setminus \Pi_2^0$.

Let X be a subset of \mathcal{N} . We let the *Cantor-Bendixson-derivative* of X , notation X' , be the set of all α in \mathcal{N} such that for each n there exists an element of X passing through $\bar{\alpha}n$, but apart from α .

Iterating this operation, we define, by induction on the set of stumps, for every subset X of \mathcal{N} and every stump σ , another subset of \mathcal{N} , called the σ -th *Cantor-Bendixson derivative* of X , notation $\text{Der}(\sigma, X)$, as follows:

- (i) $\text{Der}(\underline{1}, X) := X$
- (ii) For every non-empty stump σ , $\text{Der}(\sigma, X) := \left(\bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, X) \right)'$

Observe that for every subset X of \mathcal{N} the set X' is weakly closed.

It follows that for all subsets X of \mathcal{N} , for all stumps $\sigma < \tau$, if $\sigma < \tau$, then $\text{Der}(\tau, X) \subseteq \text{Der}(\sigma, X)$.

3.11 Lemma: Let X_0, X_1, \dots be an infinite sequence of subsets of \mathcal{N} .

Consider $Y := \{\bar{0}\} \cup \bigcup_{n \in \mathbb{N}} \bar{0}n * \langle 1 \rangle * X_n$.

- (i) $\bar{0}$ belongs to Y' if and only if there are infinitely many n such that X_n is inhabited.
- (ii) If $\bar{0}$ belongs to Y' , then Y' coincides with the closure of the set $\{\bar{0}\} \cup \bigcup_{n \in \mathbb{N}} \bar{0}n * \langle 1 \rangle * (X_n)'$.

- (iii) For each stump σ , if $\underline{0}$ belongs to $\text{Der}(\sigma, Y)$, then $\text{Der}(\sigma, Y)$ coincides with the closure of the set $\{\underline{0}\} \cup \bigcup_{n \in \mathbb{N}} \overline{0}n * \langle 1 \rangle * \text{Der}(\sigma, X_n)$.

Proof: We leave the proof to the reader. \square

3.12 Theorem: For every hereditarily repetitive stump σ , $\text{Der}(\sigma, CB_\sigma) = \emptyset$, and for every hereditarily repetitive stump τ , if $\sigma < \tau$, then $\underline{0}$ belongs to $\text{Der}(\sigma, CB_\tau)$.

Proof: We use induction on the set of stumps. Clearly $\text{Der}(\underline{1}, CB_{\underline{1}}) = CB_{\underline{1}} = \emptyset$. Also, if $\underline{1} < \tau$, then τ is non-empty and $\underline{0}$ belongs to $CB_\tau = \text{Der}(\underline{1}, CB_\tau)$.

Assume that σ is a non-empty hereditarily repetitive stump and that for each n , $\text{Der}(\sigma^n, CB_{\sigma^n}) = \emptyset$, and for each n , for every hereditarily repetitive τ , if $\sigma^n < \tau$, then $\underline{0}$ belongs to $\text{Der}(\sigma^n, CB_\tau)$.

In particular $\underline{0}$ belongs to each set $\text{Der}(\sigma^n, CB_\sigma)$.

Observe that $\bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, CB_\sigma)$ coincides with the closure of the set

$$\bigcap_{n \in \mathbb{N}} (\{\underline{0}\} \cup \bigcup_{m \in \mathbb{N}} \overline{0}m * 1 * \overline{\text{Der}(\sigma^n, CB_{\sigma^m})}) \text{ and also with the closure of the set } \{\underline{0}\} \cup \bigcup_{m \in \mathbb{N}} \overline{0}m * 1 * \bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, CB_{\sigma^m}).$$

As for each n , $\text{Der}(\sigma^n, CB_{\sigma^n}) = \emptyset$, we conclude that $\bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, CB_\sigma)$ coincides with

$$\{\underline{0}\}, \text{ and therefore } \text{Der}(\sigma, CB_\sigma) = \emptyset.$$

Now assume that τ is a hereditarily repetitive stump and $\sigma < \tau$.

Calculate m such that $\sigma \leq \tau^m$ and therefore, for every n , $\sigma^n < \tau^m$. Applying the induction hypothesis, we find that $\underline{0}$ belongs to $\bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, CB_{\tau^m})$, and therefore $\overline{0}m * \langle 1 \rangle * \underline{0}$ belongs to $\bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, CB_\tau)$. There are infinitely many numbers p such that $\tau^p = \tau^m$ and for each such p , the sequence $\overline{0}p * \langle 1 \rangle * \underline{0}$ will belong to $\bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, CB_\tau)$, therefore $\underline{0}$ is a limit point of this set, that is, $\underline{0}$ belongs to $\text{Der}(\sigma, CB_\tau)$.

We conclude that the statement of the Theorem is correct. \square

3.13 We may conclude from Theorem 3.12 that for all hereditarily repetitive stumps σ, τ , $\text{Der}(\sigma, CB_\sigma) = \emptyset$ and, if $\tau < \sigma$, then $\text{Der}(\tau, CB_\sigma) \neq \emptyset$. In this sense, σ is the least stump τ such that $\text{Der}(\tau, CB_\sigma) = \emptyset$ and might be called the *Cantor-Bendixson-rank* of CB_σ .

We now want to prove a very non-classical counterpart to Theorem 3.12.

We introduce a binary operation Perhaps on the class of subsets of \mathcal{N} . Given subsets X, Y of \mathcal{N} such that X is inhabited and X forms part of Y , we let $\text{Perhaps}(X, Y)$ be the set of all α such that there exists β in X with the property: if α is apart from β , then α belongs to Y .

The sentence “ α belongs to $\text{Perhaps}(X, Y)$ ” might be rendered into English by the

words: “ α belongs to X , perhaps (only) to Y ”. We are thinking of a speaker who interrupts himself and in doing so retracts or qualifies his earlier statement and replaces it by a second, seemingly somewhat weaker one. Observe however that we also want to reckon the statement “ α belongs to X , perhaps to X ” as a weaker one than the unconditional statement “ α belongs to X ”.

Let X be an inhabited subset of \mathcal{N} . X will be called *perhapsive* if and only if X coincides with $\text{Perhaps}(X, X)$. Frank Waaldijk, in [25], called perhapsive subsets of \mathcal{N} *weakly stable* subsets of \mathcal{N} .

3.14 Theorem:

- (i) For all subsets X, Y of \mathcal{N} ,
if X is inhabited and $X \subseteq Y$, then $X \subseteq \text{Perhaps}(X, Y) \subseteq Y^{\neg\neg}$.
- (ii) For all subsets X, Y, Z of \mathcal{N} , if X is inhabited and $X \subseteq Y \subseteq Z$, then $\text{Perhaps}(X, Y) \subseteq \text{Perhaps}(X, Z)$ and $\text{Perhaps}(X, Z) \subseteq \text{Perhaps}(Y, Z)$.
- (iii) Every inhabited Π_2^0 -subset of \mathcal{N} is perhapsive.
- (iv) For all subsets X, Y of \mathcal{N} , if Y is perhapsive and X reduces to Y , then X is perhapsive.

Proof:

(i) Suppose X is inhabited and X is a subset of Y .

Clearly, X is a subset of $\text{Perhaps}(X, Y)$.

Now assume that α belongs to $\text{Perhaps}(X, Y)$ and determine β in X such that if $\alpha \# \beta$, then α belongs to Y . If $\alpha \# \beta$, then α belongs to Y . If $\neg(\alpha \# \beta)$, then α coincides with β , and α belongs to X and therefore also to Y . As $\neg\neg(\alpha \# \beta \vee \neg(\alpha \# \beta))$, we conclude: $\neg\neg(\alpha \text{ belongs to } Y)$, that is, α belongs to $Y^{\neg\neg}$.

So $\text{Perhaps}(X, Y)$ is a subset of $Y^{\neg\neg}$.

(ii) obvious.

(iii) Suppose X is an inhabited subset of \mathcal{N} and X belongs to Π_2^0 .

Let Y_0, Y_1, \dots be a sequence of open subsets of \mathcal{N} such that $X = \bigcap_{n \in \mathbb{N}} Y_n$. Now assume

that α belongs to $\text{Perhaps}(X, X)$ and determine β in X such that, if $\alpha \# \beta$, then α belongs to X . Let n be a natural number. Determine m such that every γ passing through $\bar{\beta}m$ belongs to Y_n and distinguish two cases. Either $\bar{\alpha}m = \bar{\beta}m$, and α belongs to Y_n , or $\bar{\alpha}m \neq \bar{\beta}m$, therefore $\alpha \# \beta$, and α belongs to X and therefore in particular to Y_n . We conclude that $\text{Perhaps}(X, X)$ is a subset of every set Y_n , therefore, as X is also inhabited, $\text{Perhaps}(X, X)$ coincides with X .

(iv) Suppose X, Y are subsets of \mathcal{N} , and γ is a function from \mathcal{N} to \mathcal{N} reducing X to Y , and $\text{Perhaps}(Y, Y)$ is a subset of Y . Assume that α belongs to $\text{Perhaps}(X, X)$ and determine β in X such that, if $\alpha \# \beta$, then α belongs to X . Observe that $\gamma|\beta$ belongs to Y . Observe also that, if $\gamma|\alpha \# \gamma|\beta$, then $\alpha \# \beta$, therefore α belongs to X and $\gamma|\alpha$ belongs to Y . We conclude that $\gamma|\alpha$ belongs to $\text{Perhaps}(Y, Y)$ and so to Y , therefore α belongs to X .

We conclude that $\text{Perhaps}(X, X)$ is a subset of X .

Therefore, if also X is inhabited, $\text{Perhaps}(X, X)$ coincides with X . \square

3.15 We define, for every subset X of \mathcal{N} and every stump σ , another subset of \mathcal{N} , called *the σ -th perhapsive extension* of X , notation $\mathbb{P}(\sigma, X)$, as follows:

- (i) $\mathbb{P}(\underline{1}, X) := X$.
- (ii) For every non-empty stump σ , $\mathbb{P}(\sigma, X) := \text{Perhaps}\left(X, \bigcup_{n \in \mathbb{N}} \mathbb{P}(\sigma^n, X)\right)$

3.16 For every subset X of \mathcal{N} and every s in \mathbb{N} we let $X \upharpoonright s$ be the set of all α in X passing through s .

3.17 Lemma:

- (i) For every inhabited subset X of \mathcal{N} , for all stumps σ, τ , if $\sigma \leq \tau$, then $X \subseteq \mathbb{P}(\sigma, X) \subseteq \mathbb{P}(\tau, X) \subseteq X^{\neg\neg}$.
- (ii) For every subset X of \mathcal{N} , for every s in \mathbb{N} , for every stump σ , if $X \upharpoonright s$ is inhabited, then $\mathbb{P}(\sigma, X) \upharpoonright s = \mathbb{P}(\sigma, X \upharpoonright s)$.

Proof:

The proof is left to the reader. \square

3.18 Theorem:

- (i) For every stump σ , $\mathbb{P}(\sigma, CB_\sigma)$ coincides with $\overline{CB_\sigma} = (CB_\sigma)^{\neg\neg}$.
- (ii) For all hereditarily repetitive stumps σ, τ , if $\overline{CB_\sigma}$ forms part of $\mathbb{P}(\tau, CB_\sigma)$, then $\sigma \leq \tau$.

Proof:

- (i) Observe first that $\mathbb{P}(\underline{1}, CB_{\underline{1}})$ equals $CB_{\underline{1}} = \emptyset$ and \emptyset coincides with $\overline{\emptyset}$. Now assume that σ is a non-empty stump and, for each n , $\mathbb{P}(\sigma^n, CB_{\sigma^n})$ coincides with the closure $\overline{CB_{\sigma^n}}$ of CB_{σ^n} . We claim that the closure $\overline{CB_\sigma}$ of CB_σ coincides with $\mathbb{P}(\sigma, CB_\sigma)$, that is, with $\text{Perhaps}(CB_\sigma, \bigcup_{n \in \mathbb{N}} \mathbb{P}(\sigma^n, X))$. For suppose α belongs to $\overline{CB_\sigma}$ and α is apart from $\underline{0}$. We determine n, γ such that $\alpha = \underline{0}n * \langle 1 \rangle * \gamma$ and observe that γ belongs to $\overline{CB_{\sigma^n}}$ and therefore to $\mathbb{P}(\sigma^n, CB_{\sigma^n})$. Using Lemma 3.17(ii) we conclude that α belongs to $\mathbb{P}(\sigma^n, CB_\sigma)$. Therefore $\overline{CB_\sigma}$ forms part of and indeed coincides with $\mathbb{P}(\sigma, CB_\sigma)$.
- (ii) For each hereditarily repetitive stump τ we define the proposition $Q(\tau)$ as follows:

$$\begin{aligned} Q(\tau) &:= \tau \text{ has the property } Q \\ &:= \text{For every hereditarily repetitive stump } \sigma, \\ &\quad \text{if } \overline{CB_\sigma} \text{ forms part of } \mathbb{P}(\tau, CB_\sigma), \text{ then } \sigma \leq \tau. \end{aligned}$$

We claim that every hereditarily repetitive stump τ has the property Q . Observe first that $\underline{1}$ has the property Q , for, if, for some σ , $\overline{CB_\sigma}$ forms part of $\mathbb{P}(\underline{1}, CB_\sigma) = CB_\sigma$, then CB_σ is closed and $\sigma = \underline{1}$.

Now assume that τ is a non-empty stump and that, for each n , τ^n has the property Q . We show that τ itself has the property Q .

Suppose that σ is a hereditarily repetitive stump and that $\overline{CB_\sigma}$ forms part of $\mathbb{P}(\tau, CB_\sigma)$, that is, $\overline{CB_\sigma}$ is a subset of $\text{Perhaps}(CB_\sigma, \bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma))$. So for every α in $\overline{CB_\sigma}$ we may determine β in CB_σ such that, if α is apart from β , then α belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$. Observe that, for every β in CB_σ , we may decide $\beta = \underline{0}$ or $\beta \# \underline{0}$. So for every α in $\overline{CB_\sigma}$ there exist i, p such that there exists β with the property: *either* $i = 0$ and $\beta = \underline{0}$ *or* $i > 0$ and β passes through $\underline{0}p * \langle 1 \rangle$, and if $\alpha \# \beta$, then α belongs

to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$. Observe that $\overline{CB_\sigma}$ is a spread containing $\underline{0}$.

We apply the Continuity Principle and distinguish two cases.

First Case. We find m such that for every α in $\overline{CB_\sigma}$ passing through $\underline{0}m$, if $\alpha \# \underline{0}$, then there exists n such that α belongs to $\mathbb{P}(\tau^n, CB_\sigma)$. Consider any $p \geq m$. Observe that $\overline{CB_\sigma} \upharpoonright (\underline{0}p * \langle 1 \rangle)$ is a spread containing $\underline{0}p * \langle 1 \rangle * \underline{0}$ and forming part of $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$

and apply the Continuity Principle a second time. We find q, n such that every α in $\overline{CB_\sigma}$ passing through $\underline{0}p * \langle 1 \rangle * \underline{0}q$ belongs to $\mathbb{P}(\tau^n, CB_\sigma)$. Now observe that, for every γ , $\underline{0}p * \langle 1 \rangle * \underline{0}q * \gamma$ belongs to CB_σ , or to $\overline{CB_\sigma}$, respectively, if and only if γ itself belongs to CB_{σ^p} , or to $\overline{CB_{\sigma^p}}$, respectively. Using Lemma 3.17(ii) we conclude that for every γ , $\underline{0}p * \langle 1 \rangle * \underline{0}q * \gamma$ belongs to $\mathbb{P}(\tau^n, CB_\sigma)$ if and only if γ itself belongs to $\mathbb{P}(\tau^n, CB_{\sigma^p})$. Therefore $\overline{CB_{\sigma^p}}$ forms part of $\mathbb{P}(\tau^n, CB_{\sigma^p})$ and thus $\sigma^p \leq \tau^n$.

Clearly, for every $p \geq m$ there exists n such that $\sigma^p \leq \tau^n$, and so $\sigma \leq \tau$.

Second Case. We find m, p such that $m > p$ and for every α in $\overline{CB_\sigma}$ passing through $\underline{0}m$ there exists β in CB_σ passing through $\underline{0}p * \langle 1 \rangle$ with the property: if $\alpha \# \beta$, then there exists n such that α belongs to $\mathbb{P}(\tau^n, CB_\sigma)$. Observe that $\overline{CB_\sigma} \upharpoonright \underline{0}(m+1)$ is a spread containing $\underline{0}$ and forming part of $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$. We apply the Continuity

Principle a second time and find n, q such that $q > m$ and every α in $\overline{CB_\sigma}$ passing through $\underline{0}q$ belongs to $\mathbb{P}(\tau^n, CB_\sigma)$. Reasoning as in Case (i) we obtain the conclusion that $\overline{CB_\sigma}$ forms part of $\mathbb{P}(\tau^n, CB_\sigma)$, therefore $\sigma \leq \tau^n$ and also $\sigma \leq \tau$. \square

3.19 We may conclude from Theorem 3.18: for all hereditarily repetitive stumps σ, τ : $\sigma \leq \tau$ if and only if $\overline{CB_\sigma}$ forms part of $\mathbb{P}(\tau, CB_\sigma)$. In this sense σ is the least stump τ such that $\overline{CB_\sigma}$ forms part of $\mathbb{P}(\tau, CB_\sigma)$ and might be called the *perhapsive rank* of CB_σ .

In general given an inhabited collection C of (hereditarily repetitive) stumps, it is not possible to find σ in C such that, for all τ in C , $\sigma \leq \tau$.

3.20 Theorem:

- (i) For all subsets X, Y of \mathcal{N} , for every function γ from \mathcal{N} to \mathcal{N} , if γ maps X into Y , then for each stump σ , γ maps $\mathbb{P}(\sigma, X)$ into $\mathbb{P}(\sigma, Y)$.
- (ii) For all hereditarily stumps σ, τ , for every function γ from \mathcal{N} to \mathcal{N} , if $\sigma < \tau$ and γ maps CB_σ into CB_τ , then γ does not map surjectively the closure $\overline{CB_\sigma}$ of CB_σ onto the closure $\overline{CB_\tau}$ of CB_τ .

Proof:

(i) The proof is by induction on the set of stumps and left to the reader.

(ii) Observe that, according to (i), γ will map $\overline{CB_\sigma} = \mathbb{P}(\sigma, CB_\sigma)$ into $\mathbb{P}(\sigma, CB_\tau)$ and, according to Theorem 3.18, the latter set is a proper subset of $\overline{CB_\tau}$. \square

4 Perhaps and Almost

We continue the study of the notion Perhaps discovered in the previous Section and introduce a further notion: Almost. We also consider countable sets that are dense in \mathcal{N} .

4.1 Let X be a subset of \mathcal{N} . We let $\text{Almost}(X)$ be the set of all α in \mathcal{N} such that for some stump σ , α belongs to $\mathbb{P}(\sigma, X)$. Observe that this definition involves a quantification on the set **Stp** of stumps. The possibility of introducing the set $\text{Almost}(X)$ depends upon our acceptance of **Stp** as a set and a domain of quantification.

4.2 Theorem:

- (i) For all inhabited subsets X of \mathcal{N} $X \subseteq \text{Almost}(X) \subseteq X^{\neg\neg}$.
- (ii) For all inhabited subsets X of \mathcal{N} , $\text{Perhaps}(X, \text{Almost}(X))$ coincides with $\text{Almost}(X)$.
- (iii) For all subsets X, Y, Z of \mathcal{N} , if X is inhabited and $X \subseteq Y \subseteq Z$, then $\text{Perhaps}(\text{Perhaps}(X, Y), Z)$ forms part of $\text{Perhaps}(X, \text{Perhaps}(Y, Z))$.
- (iv) For all inhabited subsets X of \mathcal{N} , for every stump σ , $\text{Perhaps}(\mathbb{P}(\sigma, X), \text{Almost}(X))$ forms part of $\text{Almost}(X)$.
- (v) For all inhabited subsets X of \mathcal{N} , $\text{Perhaps}(\text{Almost}(X), \text{Almost}(X))$ coincides with $\text{Almost}(X)$, that is, $\text{Almost}(X)$ is perhapsive.
- (vi) For all inhabited subsets X, Y of \mathcal{N} , if $X \subseteq Y$ and Y is perhapsive, then $\text{Almost}(X)$ forms part of Y .

Proof:

(i) is an almost direct consequence of Lemma 3.17(i).

(ii) Let X be a subset of \mathcal{N} and suppose that α belongs to $\text{Perhaps}(X, \text{Almost}(X))$. Find β in X such that, if α is apart from β , then α belongs to $\text{Almost}(X)$. We now build a non-empty stump σ by specifying successively its immediate substumps $\sigma^0, \sigma^1, \sigma^2, \dots$. For each n , if $\alpha(n) = \beta(n)$ or if there exists $p < n$ such that $\alpha(p) \neq \beta(p)$ then σ^n is the empty stump $\underline{1}$, and if $\alpha(n) \neq \beta(n)$ and there is no $p < n$ such that $\alpha(p) \neq \beta(p)$, we find a stump τ such that α belongs to $\text{Perhaps}(\tau, X)$, and define $\sigma^n := \tau$. We claim that α belongs to $\mathbb{P}(\sigma, X)$. For let β be the sequence we just considered and observe: β belongs to X , and if α is apart from β , and n is the least p such that $\alpha(p) \neq \beta(p)$, then α belongs to $\mathbb{P}(\sigma^n, X)$.

We conclude that α belongs to $\text{Almost}(X)$.

Therefore $\text{Perhaps}(X, \text{Almost}(X))$ is part of $\text{Almost}(X)$ and, if X is inhabited, the two sets coincide.

(iii) Let X, Y, Z be subsets of \mathcal{N} such that X is inhabited and $X \subseteq Y \subseteq Z$. Assume that α belongs to $\text{Perhaps}(\text{Perhaps}(X, Y), Z)$. We intend to show that α belongs to $\text{Perhaps}(X, \text{Perhaps}(Y, Z))$. First determine β in $\text{Perhaps}(X, Y)$ such that if α is apart from β , then α belongs to Z . Then determine γ in X such that if β is apart

from γ , then β belongs to Y . Now assume that α is apart from γ and distinguish two cases: *either* α is apart from β and α belongs to Z , *or* γ is apart from β and β belongs to Y , and therefore α belongs to $\text{Perhaps}(Y, Z)$. In both cases α belongs to $\text{Perhaps}(Y, Z)$, so if α is apart from γ , then α belongs to $\text{Perhaps}(Y, Z)$, therefore α belongs to $\text{Perhaps}(X, \text{Perhaps}(Y, Z))$.

We conclude that $\text{Perhaps}(\text{Perhaps}(X, Y), Z)$ forms part of $\text{Perhaps}(X, \text{Perhaps}(Y, Z))$.

(iv) Let X be an inhabited subset of \mathcal{N} . We claim that for each stump σ , the set $\text{Perhaps}(\mathbb{P}(\sigma, X), \text{Almost}(X))$ forms part of $\text{Almost}(X)$.

We use induction on the set of stumps.

We know from (ii) that $\text{Perhaps}(\mathbb{P}(1, X), \text{Almost}(X))$ forms part of $\text{Almost}(X)$.

Now assume that σ is a non-empty stump and that, for each n ,

$\text{Perhaps}(\mathbb{P}(\sigma^n, X), \text{Almost}(X))$ forms part of $\text{Almost}(X)$.

Consider $\text{Perhaps}(\mathbb{P}(\sigma, X), \text{Almost}(X))$, observe that $\mathbb{P}(\sigma, X)$ coincides with $\text{Perhaps}(X, \bigcup_{n \in \mathbb{N}} \mathbb{P}(\sigma^n, X))$ and apply (iii) in order to conclude:

$\mathbb{P}(\sigma, X)$ forms part of $\text{Perhaps}\left(X, \text{Perhaps}\left(\bigcup_{n \in \mathbb{N}} \mathbb{P}(\sigma^n, X), \text{Almost}(X)\right)\right)$ and $\text{Perhaps}\left(\bigcup_{n \in \mathbb{N}} \mathbb{P}(\sigma^n, X), \text{Almost}(X)\right)$ coincides with

$\bigcup_{n \in \mathbb{N}} \text{Perhaps}(\mathbb{P}(\sigma^n, X), \text{Almost}(X))$ and therefore with $\text{Almost}(X)$.

Therefore, $\mathbb{P}(\sigma, X)$ forms part of $\text{Perhaps}(\mathbb{P}(\sigma, X), \text{Almost}(X))$.

(v) Let X be an inhabited subset of \mathcal{N} .

Observe that $\text{Perhaps}(\text{Almost}(X), \text{Almost}(X))$ coincides with

$\text{Perhaps}\left(\bigcup_{\sigma \in \mathbf{Stp}} \mathbb{P}(\sigma, X), \text{Almost}(X)\right)$ and also with

$\bigcup_{\sigma \in \mathbf{Stp}} \text{Perhaps}(\mathbb{P}(\sigma, X), \text{Almost}(X))$, and therefore, according to (iv), with $\text{Almost}(X)$.

(vi) Let X be inhabited subset of \mathcal{N} , and suppose Y is a *perhapsive* subset of \mathcal{N} containing X . One may prove by induction on the set of stumps that for each stump σ , $\mathbb{P}(\sigma, X)$ forms part of Y , and therefore $\text{Almost}(X)$ forms part of Y . \square

4.3 Let X be an inhabited subset of \mathcal{N} .

In view of Theorem 4.2 we may call $\text{Almost}(X)$ the *perhapsive closure* of the set X : $\text{Almost}(X)$ is the least *perhapsive* set containing X .

4.4 Let D be a subset of \mathcal{N} . D is *dense-in-itself* if and only if for every α in D , for every m , there exists β in D such that β is apart from α and $\overline{\alpha}m = \overline{\beta}m$. D is *discrete* if and only if for all α, β in D we may decide if $\alpha \# \beta$ or $\alpha = \beta$.

Let X be a subset of \mathcal{N} and Y a subset of X . Y is a *decidable* subset of X if and only if we may decide, for every α in X , if α belongs to Y or not.

4.5 Theorem:

Let D be an enumerable and discrete subset of \mathcal{N} that is also dense-in-itself.

- (i) For each stump σ , for every s , if s contains an element of D , then there exists an embedding of $\overline{CB_\sigma}$ into $\mathcal{N} \upharpoonright s$ mapping CB_σ itself onto a decidable subset of D .
- (ii) For each stump σ , the set CB_σ reduces to the set D .
- (iii) For no stump σ , D reduces to CB_σ .
- (iv) For each stump σ , for each stump τ , the set $\mathbb{P}(\tau, CB_\sigma)$ reduces to the set $\mathbb{P}(\tau, D)$.
- (v) For all hereditarily repetitive stumps σ, τ , if $\sigma < \tau$, then $\mathbb{P}(\sigma, D)$ is a proper subset of the set $\mathbb{P}(\tau, D)$.

Proof:

(i) We use induction on the set of stumps.

The statement is obviously true in case σ equals the empty stump $\underline{1}$. Now assume σ is a non-empty stump and for each n, s such that s contains an element of D there exists an embedding of $\overline{CB_{\sigma^n}}$ into $\mathcal{N} \upharpoonright s$ mapping CB_{σ^n} onto a decidable subset of D . Let s be a natural number containing an element of D . Find α in \mathcal{N} such that $s * \alpha$ belongs to D . We calculate two sequences k_0, k_1, \dots and p_0, p_1, \dots of natural numbers such that $k_0 < k_1 < \dots$ and, for each n , p_n is different from $\alpha(k_n)$ and $s * \overline{\alpha}k_n * \langle p_n \rangle$ contains an element of D . For each n we construct an embedding δ_n of CB_{σ^n} into $\mathcal{N} \upharpoonright s * \overline{\alpha}k_n * \langle p_n \rangle$ mapping CB_{σ^n} onto a decidable subset of D . It is not difficult to define an embedding γ from $\overline{CB_\sigma}$ into $\mathcal{N} \upharpoonright s$ such that $\gamma|_{\underline{0}}$ equals $s * \alpha$ and for each n , for each β in $\overline{CB_{\sigma^n}}$, $\gamma|_{\underline{0}n * \langle 1 \rangle * \beta}$ equals $s * \overline{\alpha}k_n * \langle p_n \rangle * \delta_n|_{\beta}$. Observe that γ maps CB_σ itself onto a decidable subset of D .

(ii) Let σ be a stump and, using (i), define an embedding γ from $\overline{CB_\sigma}$ into \mathcal{N} mapping CB_σ itself onto a decidable subset of D . Let δ be an enumeration of D , that is, D coincides with the set $\{\delta^0, \delta^1, \dots\}$. Let E be the set of all s in \mathbb{N} that contain an element of CB_σ .

Observe that E is a decidable subset of \mathbb{N} . We may construct a function ζ from \mathcal{N} to \mathcal{N} such that for all α in $\overline{CB_\sigma}$, $\zeta|_{\alpha}$ coincides with $\gamma|_{\alpha}$, and for each α, n , if $\overline{\alpha}n$ does not belong to E , then for each i , $(\zeta|_{\alpha})(n+i)$ differs from $\delta^i(n+i)$.

We claim that ζ reduces CB_σ to D .

It will be clear that for every α , if α belongs to CB_σ , then $\zeta|_{\alpha}$ belongs to D . Assume now that α is an element of \mathcal{N} and $\zeta|_{\alpha}$ belongs to D . Then every initial part of α belongs to E and α belongs to $\overline{CB_\sigma}$ and $\zeta|_{\alpha}$ coincides with $\gamma|_{\alpha}$. We may decide if there exists β in CB_σ such that $\gamma|_{\beta}$ equals $\gamma|_{\alpha}$. As $CB_\sigma^{\neg\top}$ coincides with $\overline{CB_\sigma}$, it is excluded that α does not belong to CB_σ , so there exists β in CB_σ such that $\gamma|_{\beta}$ equals $\gamma|_{\alpha}$. As γ is an embedding, we must have $\alpha = \beta$, that is, α belongs to CB_σ .

This completes the proof of our claim that ζ reduces CB_σ to D .

(iii) is an easy consequence of (ii) and Theorem 3.9(ii).

(iv) Let σ be a stump and let ζ be a function from \mathcal{N} to \mathcal{N} embedding CB_σ onto a decidable subset of D and reducing CB_σ to D . We constructed such a function in (ii). We claim that for each stump τ , ζ reduces the set $\mathbb{P}(\tau, CB_\sigma)$ to the set $\mathbb{P}(\tau, D)$.

and prove this claim by induction on the set of stumps. The statement is obviously true if τ equals the empty stump $\underline{1}$. Now assume that τ is a non-empty stump and, for each n , ζ reduces the set $\mathbb{P}(\tau^n, CB_\sigma)$ to the set $\mathbb{P}(\tau^n, D)$. Suppose that α belongs to $\mathbb{P}(\tau, CB_\sigma)$. Find β in CB_σ such that, if α is apart from β , then α belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$. Observe that, if $\zeta|\alpha$ is apart from $\zeta|\beta$, then α is apart from β , and α belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$, and $\zeta|\alpha$ belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, D)$. As $\zeta|\beta$ belongs to D , this shows that ζ maps $\mathbb{P}(\tau, CB_\sigma)$ into $\mathbb{P}(\tau, D)$.

Now assume that α belongs to \mathcal{N} and $\zeta|\alpha$ belongs to $\mathbb{P}(\tau, D)$. Find β in D such that, if $\zeta|\alpha$ is apart from β , then $\zeta|\alpha$ belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, D)$. Now find out if there exists δ in CB_σ such that $\zeta|\delta$ equals β and distinguish two cases.

First Case. There exists δ in CB_σ such that $\zeta|\delta$ equals β , say δ_0 . Observe that, if α is apart from δ_0 , then $\zeta|\alpha$ is apart from $\zeta|\delta_0 = \beta$, therefore $\zeta|\alpha$ belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, D)$ and α belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$. As δ_0 belongs to CB_σ , this shows that α belongs to $\mathbb{P}(\tau, CB_\sigma)$.

Second Case. There is no δ in CB_σ such that $\zeta|\delta$ equals β . As β belongs to D and D is discrete, this implies that, for every δ in CB_σ , β is apart from $\zeta|\delta$. It follows that, for every δ in the closure $\overline{CB_\sigma}$ of CB_σ , β is apart from $\zeta|\delta$.

Now observe that $\zeta|\alpha$ belongs to $\mathbb{P}(\tau, D)$ and therefore to $D^{\neg\neg}$, so α itself belongs to $CB_\sigma^{\neg\neg} = \overline{CB_\sigma}$, therefore β is apart from $\zeta|\alpha$, and $\zeta|\alpha$ belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, D)$ and α itself belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, CB_\sigma)$, and therefore also to $\mathbb{P}(\tau, CB_\sigma)$.

We have shown, for every α , if $\zeta|\alpha$ belongs to $\mathbb{P}(\tau, D)$, then α belongs to $\mathbb{P}(\tau, CB_\sigma)$ and conclude: ζ reduces $\mathbb{P}(\tau, CB_\sigma)$ to $\mathbb{P}(\tau, D)$.

(v) Let σ, τ be hereditarily repetitive stumps such that $\sigma < \tau$.

We have seen, in Theorem 3.18 that $\mathbb{P}(\tau, CB_\tau)$ coincides with $\overline{CB_\tau}$ and $\mathbb{P}(\sigma, CB_\tau)$ does not, and therefore, in view of Lemma 3.17, $\mathbb{P}(\sigma, CB_\tau)$ is a proper subset of $\mathbb{P}(\tau, CB_\tau)$.

Let ζ be a function from \mathcal{N} to \mathcal{N} reducing CB_τ to D and also $\mathbb{P}(\sigma, CB_\tau)$ to $\mathbb{P}(\sigma, D)$ and $\mathbb{P}(\tau, CB_\tau)$ to $\mathbb{P}(\tau, D)$. (We know from our proof of (iv) that there exists such a function). Assume that $\mathbb{P}(\tau, D)$ is a subset of $\mathbb{P}(\sigma, D)$. It follows that $\mathbb{P}(\tau, CB_\tau)$ is a subset of $\mathbb{P}(\sigma, CB_\tau)$ and we obtain a contradiction.

We conclude that $\mathbb{P}(\tau, D)$ is not a subset of $\mathbb{P}(\sigma, D)$, therefore, $\mathbb{P}(\sigma, D)$ is a proper subset of $\mathbb{P}(\tau, D)$. \square

4.6 Let X be an inhabited subset of \mathcal{N} . X has *unbounded perhapsity* if and only if for all hereditarily repetitive stumps σ, τ , if $\sigma < \tau$, then $\mathbb{P}(\sigma, X)$ is a proper subset of $\mathbb{P}(\tau, X)$. Theorem 4.5(v) is the statement that every enumerable and discrete subset of \mathcal{N} that is also dense-in-itself, has unbounded perhapsity. Let X be a non-empty enumerable subset of \mathcal{N} and let δ be an enumeration of X , that is, $X = \{\delta^0, \delta^1, \dots\}$.

We let $\text{Almost}^*(X)$ be the set of all α in \mathcal{N} such that for each β there exists n such that $\overline{\alpha}(\beta(n)) = \overline{\delta^n}(\beta(n))$. Intuitively spoken, the set $\text{Almost}^*(X)$ is not that much different from the set X . We could perhaps say that $\text{Almost}^*(X)$ is the set of all α in \mathcal{N} for which we see that every attempt to prove that α is apart from every element in the sequence $\delta^0, \delta^1, \dots$ will fail in finitely many steps. Every β in \mathcal{N} might be thought of as expressing the hope that, for all n , $\overline{\alpha}(\beta(n)) \neq \overline{\delta^n}(\beta(n))$.

4.7 Theorem:

Let X be an enumerable subset of \mathcal{N} .

- (i) $X \subseteq \text{Almost}^*(X)$
- (ii) $\text{Perhaps}(X, \text{Almost}^*(X)) = \text{Almost}^*(X)$
- (iii) $\text{Almost}^*(X)$ is perhapsive, that is,
 $\text{Perhaps}(\text{Almost}^*(X), \text{Almost}^*(X)) = \text{Almost}^*(X)$.
- (iv) $\text{Almost}(X) \subseteq \text{Almost}^*(X)$

Proof:

(i) obvious.

(ii) Let δ be an enumeration of X , so $X = \{\delta^0, \delta^1, \dots\}$.

Assume that α belongs to $\text{Perhaps}(X, \text{Almost}^*(X))$, and let δ^n be an element of X such that, if α is apart from δ^n , then α belongs to $\text{Almost}^*(X)$.

Let β belong to \mathcal{N} and distinguish two cases.

Either $\overline{\alpha}(\beta(n))$ equals $\overline{\delta^n}(\beta(n))$ *or* $\overline{\alpha}(\beta(n))$ is different from $\overline{\delta^n}(\beta(n))$. In the latter case α is apart from δ^n , therefore α belongs to $\text{Almost}^*(X)$ and there exists m such that $\overline{\alpha}(\beta(m))$ equals $\overline{\delta^m}(\beta(m))$.

So in both cases there exists m such that $\overline{\alpha}(\beta(m)) = \overline{\delta^m}(\beta(m))$.

We conclude that α belongs to $\text{Almost}^*(X)$.

(iii) Let δ be an enumeration of X , so $X = \{\delta^0, \delta^1, \dots\}$.

Assume that α belongs to $\text{Perhaps}(\text{Almost}^*(X), \text{Almost}^*(X))$ and let γ be an element of $\text{Almost}^*(X)$ such that, if α is apart from γ , then α belongs to $\text{Almost}^*(X)$. Let β belong to \mathcal{N} and find n such that $\overline{\gamma}(\beta(n)) = \overline{\delta^n}(\beta(n))$, and distinguish two cases.

Either $\overline{\alpha}(\beta(n))$ equals $\overline{\gamma}(\beta(n))$ *or* $\overline{\alpha}(\beta(n))$ is different from $\overline{\gamma}(\beta(n))$. In the latter case α is apart from γ , therefore α belongs to $\text{Almost}^*(X)$, and there exists m such that $\overline{\alpha}(\beta(m)) = \overline{\delta^m}(\beta(m))$. So in both cases there exists m such that $\overline{\alpha}(\beta(m)) = \overline{\delta^m}(\beta(m))$.

We conclude that α belongs to $\text{Almost}^*(X)$.

(iv) This follows from (iii) and Theorem 4.2(vi). \square

4.8 For every δ , we let D_δ be the enumerable set $\{\delta^0, \delta^1, \dots\}$.

Let σ be a stump, and suppose δ, α belong to \mathcal{N} .

σ secures α with respect to δ if and only if for every β there exists n such that $\overline{\beta n}$

belongs to σ and there exists $m < n$ such that $\overline{\alpha}(\beta(m)) = \overline{\delta^m}(\beta(m))$.
The following statement follows from Brouwer's Thesis:

For every δ, α , if α belongs to $\text{Almost}^*(D_\delta)$,
then there exists a stump σ that secures α with respect to δ .

4.9 Theorem:

- (i) For every stump σ , for every α , for every δ ,
if σ secures α with respect to δ , then α belongs to $\mathbb{P}(\sigma, D_\delta)$
- (ii) (Using *Brouwer's Thesis*)
For every enumerable subset X of \mathcal{N} ,
 $\text{Almost}^*(X) \subseteq \text{Almost}(X)$ and $\text{Almost}^*(X) \subseteq X^{\neg\neg}$.

Proof: (i) For every stump σ we define the proposition $P(\sigma)$ as follows:

$$\begin{aligned} P(\sigma) &:= \sigma \text{ has the property } P \\ &:= \text{For all } \alpha, \delta, \text{ if } \sigma \text{ secures } \alpha \text{ with respect to } \delta, \\ &\quad \text{then } \alpha \text{ belongs to } \mathbb{P}(\sigma, D_\delta). \end{aligned}$$

We prove, by induction on the set of stumps, that every stump has the property P .
Observe that the empty stump $\underline{1}$ has the property P , as there are no α, δ such that $\underline{1}$ secures α with respect to δ .

Now assume that σ is a non-empty stump and that for each n , σ^n has the property P . We show that σ itself has the property P . Suppose α, δ are such that σ secures α with respect to δ . Assume α is apart from δ^0 and find n such that $\overline{\alpha}(n) \neq \overline{\delta^0}n$. Define ζ in \mathcal{N} such that, for each i , $\zeta^i = \delta^{i+1}$ and observe that σ^n secures α with respect to ζ . Therefore α belongs to $\mathbb{P}(\sigma^n, D_\zeta)$ and, as D_ζ is a subset of D_δ , also to $\mathbb{P}(\sigma^n, D_\delta)$. Therefore, if α is apart from δ^0 , there exists n such that α belongs to $\mathbb{P}(\sigma^n, D_\delta)$. As δ^0 belongs to D_δ , α belongs to $\mathbb{P}(\sigma, D_\delta)$.

We conclude that σ has the property P .

(ii) follows easily from (i) and the remark in Section 4.8. □

4.10 One of the results in [18] may be seen to assert that $\text{Almost}(E_2)$ does not coincide with E_2 .

We let **Fin** be the set of all α in Cantor space \mathcal{C} such that there exists n such that, for all $j > n$, $\alpha(j) = 0$. An element α of \mathcal{C} belongs to **Fin** if and only if α is the characteristic function of a finite subset of the set \mathbb{N} of natural numbers.

Fin is an enumerable and discrete subset of \mathcal{N} that is dense-in-itself. For each stump σ , CB_σ is a decidable subset of **Fin**.

According to Theorem 4.5, for every stump σ , CB_σ reduces to **Fin**.

We considered the set $\text{Almost}^*(\mathbf{Fin})$ in [20] en [21] and observed, that upon the additional assumption of the generalized form of Markov's Principle, the set $\text{Almost}^*(\mathbf{Fin})$ coincides with the set $\mathbf{Fin}^{\neg\neg}$.

5 Finite unions of closed sets

We introduce a binary operation called *disjunction* on the class of subsets of \mathcal{N} and study some of its properties. We show that there are uncountably many sets X such that $CB_{2*} \subseteq X \subseteq CB_{2*}^{\neg}$. Thereafter, we bring in *conjunction*, and we study finite intersections of finite unions of closed sets.

5.1 We let **Inf** be the set of all α in Cantor space \mathcal{C} such that for each n there exists $j > n$ such that $\alpha(j) = 0$.

An element α of \mathcal{N} belongs to **Inf** if and only if α is the characteristic function of an infinite subset of the set of natural numbers. **Inf** is a countable intersection of open sets and thus belongs to the class Π_2^0 .

5.2 Theorem:

Every Π_2^0 -subset of \mathcal{N} reduces to **Inf**.

Proof:

Let X be a Π_2^0 -subset of \mathcal{N} and assume that Y_0, Y_1, \dots is a sequence of open subsets of \mathcal{N} such that $X = \bigcap_{n \in \mathbb{N}} Y_n$. Let C_0, C_1, \dots be a sequence of decidable subsets of \mathbb{N} such that for each α , for each n , α belongs to Y_n if and only if some initial part of α belongs to C_n . We define a function γ from \mathcal{N} to \mathcal{N} such that for each α , for each n , $(\gamma|\alpha)(n)$ belongs to $\{0, 1\}$ and $(\gamma|\alpha)(n) = 1$ if and only if the least $i < n + 1$ such that for every $j < i$ some initial part of $\bar{\alpha}(n + 1)$ belongs to C_j is greater than the least $i < n$ such that for every $j < i$ some initial part of $\bar{\alpha}n$ belongs to C_j .

One verifies without difficulty that γ reduces X to **Inf**. \square

5.3 We shall see soon that the set **Fin** is not a complete element of the class Σ_2^0 , and thus thwart an expectation one might form after Theorem 5.2.

We define a binary operation D on the class of subsets of Baire space \mathcal{N} . For all subsets X, Y of \mathcal{N} we let $D(X, Y)$ be the set of all α such that either α^0 belongs to X or α^1 belongs to Y . We call the set $D(X, Y)$ the *disjunction* of the sets X and Y . Observe that, for all subsets X, Y, Z of \mathcal{N} , Z reduces to $D(X, Y)$ if and only if there exist subsets Z_0, Z_1 of \mathcal{N} such that $Z = Z_0 \cup Z_1$ and Z_0 reduces to X and Z_1 reduces to Y .

For every subset X of \mathcal{N} we denote $D(X, X)$ by $D^2(X)$.

We define a subset A_1 of \mathcal{N} . A_1 is the set of all α such that, for every n , $\alpha(n) = 0$. So the sequence $\underline{0}$ is the one and only element of A_1 .

Observe that, for every subset X of \mathcal{N} , X reduces to A_1 if and only if X is closed and X reduces to $D^2(A_1)$ if and only if there exist closed sets X_0, X_1 such that $X = X_0 \cup X_1$.

Observe that the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ is a spread containing $\underline{0}$.

5.4 Theorem:

- (i) $D^2(A_1)$ is not closed
- (ii) The closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ coincides with $\text{Perhaps}(D^2(A_1), D^2(A_1))$ and with $D^2(A_1)^{\neg\neg}$.
- (iii) $D^2(A_1)$ is not perhapsive and does not belong to $\mathbf{\Pi}_2^0$.
- (iv) $D^2(A_1)$ belongs to $\mathbf{\Sigma}_2^0$ but does not reduce to **Fin**.
- (v) **Fin** does not reduce to $D^2(A_1)$.

Proof:

(i) Suppose that $\overline{D^2(A_1)}$ coincides with $D^2(A_1)$. For every α in the spread $\overline{D^2(A_1)}$ we may decide either $\alpha^0 = \underline{0}$ or $\alpha^1 = \underline{0}$. Applying the Continuity Principle we find m such that either for every α in $\overline{D^2(A_1)}$ passing through $\underline{0}m$, $\alpha^0 = \underline{0}$ or for every α in $\overline{D^2(A_1)}$ passing through $\underline{0}m$, $\alpha^1 = \underline{0}$.

This is absurd, as for each m there exist α, β in $D^2(A_1)$ passing through $\underline{0}m$ such that α^0 is apart from $\underline{0}$ and β^1 is apart from $\underline{0}$.

We conclude that $\overline{D^2(A_1)}$ is not a subset of $D^2(A_1)$.

(ii) It will be clear that $\text{Perhaps}(D^2(A_1), D^2(A_1))$ forms part of $D^2(A_1)^{\neg\neg}$ and that $D^2(A_1)^{\neg\neg}$ is included in $\overline{D^2(A_1)}$. We now show that $\overline{D^2(A_1)}$ is a subset of $\text{Perhaps}(D^2(A_1), D^2(A_1))$. Assume that α belongs to $\overline{D^2(A_1)}$. Define β in \mathcal{N} such that $\beta^0 = \underline{0}$ and for each n , if there exists no p such that $n = \langle 0, p \rangle$, then $\beta(n) = \alpha(n)$. Observe that β belongs to $D^2(A_1)$, and, if α is apart from β , then α^0 is apart from $\underline{0}$ and consequently α^1 coincides with $\underline{0}$, so α belongs to $D^2(A_1)$.

Therefore, α belongs to $\text{Perhaps}(D^2(A_1), D^2(A_1))$.

(iii) follows from (ii) and Theorem 3.14(iii).

(iv) $D^2(A_1)$ obviously belongs to $\mathbf{\Sigma}_2^0$. Assume now that γ is a function from \mathcal{N} to \mathcal{N} reducing $D^2(A_1)$ to **Fin**. Let B_0 be the set of all α in \mathcal{N} such that $\alpha^0 = \underline{0}$ and let B_1 be the set of all α in \mathcal{N} such that $\alpha^1 = \underline{0}$. Observe that B_0, B_1 are spreads and that $D^2(A_1) = B_0 \cup B_1$. For every α in $D^2(A_1)$ there exists m such that, for every $i > m$, $(\gamma|\alpha)(i) = 0$. Applying the Continuity Principle two times, we find n, m such that for every α from $B_0 \cup B_1$, if $\bar{\alpha}n = \underline{0}n$ then, for every $i > m$, $(\gamma|\alpha)(i) = 0$. Observe that now, also for every α from the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$, for every $i > m$, $(\gamma|\alpha)(i) = 0$, so $\gamma|\alpha$ belongs to **Fin**. Therefore $\overline{D^2(A_1)}$ is part of $D^2(A_1)$, and this contradicts (i).

(v) Let γ be a function from \mathcal{N} to \mathcal{N} reducing **Fin** to $D^2(A_1)$. As, for each m , $\bar{1}m * \underline{0}$ belongs to **Fin**, $\gamma|\bar{1}$ will belong to the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ and therefore, in view of (ii), to $D^2(A_1)^{\neg\neg}$. So $\bar{1}$ belongs to **Fin** $^{\neg\neg}$. But $\bar{1}$ does not belong to **Fin**. \square

5.5 Let X be a subset of \mathcal{N} and n a positive natural number.

We define a subset of \mathcal{N} , the n -fold disjunction of X , notation $D^n(X)$. $D^n(X)$ is the set of all α in \mathcal{N} such that, for some $k < n$, α^k belongs to X .

Observe that, for every subset Z of \mathcal{N} , Z reduces to $D^n(X)$ if and only if there exist subsets Z_0, Z_1, \dots, Z_{n-1} of \mathcal{N} , each of them reducing to X , such that $Z = Z_0 \cup Z_1 \cup \dots \cup Z_{n-1}$. It is easily seen that for each subset X of \mathcal{N} , for every positive

n , $D^n(X)$ reduces to $D^{n+1}(X)$.

Observe that, for each positive n , the closure $\overline{D^n(A_1)}$ of $D^n(A_1)$ is a spread containing $\underline{0}$. For every α , for each positive n , α belongs to $\overline{D^n(A_1)}$ if and only if for each k , the sequence $\underline{0}$ passes through one of $\overline{\alpha^0 k}, \overline{\alpha^1 k}, \dots, \overline{\alpha^{n-1} k}$.

Recall that we defined a special sequence $0^*, 1^*, \dots$ of stumps in Section 1.5.1. Observe that for every subset X of \mathcal{N} , for every n , $\mathbb{P}((n+1)^*, X)$ equals $\text{Perhaps}(X, \mathbb{P}(n^*, X))$, and $\mathbb{P}(0^*, X) = X$.

5.6 Theorem:

- (i) For each n , the closure $\overline{D^{n+1}(A_1)}$ of $D^{n+1}(A_1)$ coincides with $\mathbb{P}(n^*, D^{n+1}(A_1))$.
- (ii) For each n , for each stump σ , if $\overline{D^{n+1}(A_1)}$ coincides with $\mathbb{P}(\sigma, D^{n+1}(A_1))$, then $n^* \leq \sigma$.
- (iii) For each n , for each function γ from \mathcal{N} to \mathcal{N} , if γ maps $D^n(A_1)$ into $D^{n+1}(A_1)$, then γ does not map surjectively the closure $\overline{D^n(A_1)}$ of $D^n(A_1)$ onto the closure $\overline{D^{n+1}(A_1)}$ of $D^{n+1}(A_1)$.
- (iv) For each positive n , the set $D^{n+1}(A_1)$ does not reduce to the set $D^n(A_1)$.

Proof:

(i) We use induction. Observe that $\mathbb{P}(0^*, D^1(A_1))$ coincides with $D^1(A_1)$, and $D^1(A_1)$ coincides with its closure $\overline{D^1(A_1)}$.

Let n be a natural number and assume $\overline{D^{n+1}(A_1)}$ coincides with $\mathbb{P}(n^*, D^{n+1}(A_1))$. Suppose that α belongs to $\overline{D^{n+2}(A_1)}$. Define β such that $\beta^{n+1} = \underline{0}$ and for each p , if there does not exist i such that $p = \langle n+1, i \rangle$, then $\beta(p) = \alpha(p)$. Observe that β belongs to $D^{n+2}(A_1)$, and if $\alpha \# \beta$, then $\alpha^{n+1} \# \underline{0}$ and α belongs to $\overline{D^{n+1}(A_1)}$ and so to $\mathbb{P}(n^*, D^{n+1}(A_1))$ and $\mathbb{P}(n^*, D^{n+1}(A_1))$ is a subset of $\mathbb{P}(n^*, D^{n+2}(A_1))$.

Therefore α belongs to $\mathbb{P}((n+1)^*, D^{n+2}(A_1))$.

This shows that $\overline{D^{n+2}(A_1)}$ forms part of and indeed coincides with $\mathbb{P}((n+1)^*, D^{n+2}(A_1))$.

(ii) We use induction. The statement to be proven is trivially true if $n = 0$. Now let n be a natural number and assume that, for each stump σ , if $\overline{D^{n+1}(A_1)}$ coincides with $\mathbb{P}(\sigma, D^{n+1}(A_1))$, then $n^* \leq \sigma$. Let σ be a stump such that $\overline{D^{n+2}(A_1)}$ coincides with $\mathbb{P}(\sigma, D^{n+2}(A_1))$. For each α in $\overline{D^{n+2}(A_1)}$ we can find β in $D^{n+2}(A_1)$ such that, if $\alpha \# \beta$, then α belongs to

$\bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+2}(A_1))$. For every β in $D^{n+2}(A_1)$ we can find $i < n+2$ such that $\beta^i = \underline{0}$. $\overline{D^{n+2}(A_1)}$ is a spread containing $\underline{0}$. We apply the Continuity Principle and find m, i such that $i < n+2$ and for all α in $\overline{D^{n+2}(A_1)}$ passing through $\underline{0}m$ there exists β such that $\beta^i = \underline{0}$ and, if $\alpha \# \beta$, then α belongs to $\bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+2}(A_1))$. Without

endangering generality, we may assume $i = n+1$. Now consider the set B of all α in $\overline{D^{n+2}(A_1)}$ such that $\overline{\alpha}m = \underline{0}m$ and $\alpha^{n+1} = \underline{0}m * \underline{1}$. Observe that B is a spread

and forms part of $\bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+2}(A_1))$. As for each α in B , $\alpha^{n+1} \# \underline{0}$, one may prove, for every stump τ , for each α in B , if α belongs to $\mathbb{P}(\tau, D^{n+2}(A_1))$, then α belongs to $\mathbb{P}(\tau, D^{n+1}(A_1))$. So B forms part of $\bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1))$. In fact, the set of all α in $\overline{D^{n+1}(A_1)}$ passing through $\underline{0}m$ forms part of $\bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1))$. (Given any such α , consider β in B such that for every $i < n+2$, $\beta^i = \alpha^i$ and observe that β belongs to $\bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1))$ and therefore also α does so.) The set $\overline{D^{n+1}(A_1)}$ is a spread containing $\underline{0}$. We apply the Continuity Principle a second time and find q, k such that every α in $\overline{D^{n+1}(A_1)}$ passing through $\underline{0}k$ belongs to $\mathbb{P}(\sigma^k, D^{n+1}(A_1))$. It is not difficult to see that now also $\overline{D^{n+1}(A_1)}$ forms part of $\mathbb{P}(\sigma^k, D^{n+1}(A_1))$, therefore $n^* \leq \sigma^k$ and $(n+1)^* \leq \sigma$.

(iii) The argument is similar to the argument for Theorem 3.20 and left to the reader.

(iv) Let n be a positive natural number and suppose γ is a function from \mathcal{N} to \mathcal{N} reducing $D^{n+1}(A_1)$ to $D^n(A_1)$. For each $i < n+1$, let B_i be the set of all α such that $\alpha^i = \underline{0}$. Observe that each B_i is a spread containing $\underline{0}$ and that γ maps $\bigcup_{i < n+1} B_i$ into $\bigcup_{i < n} B_i$. Applying the Continuity Principle $n+1$ times we find natural numbers p_0, p_1, \dots, p_n and k_0, k_1, \dots, k_n such that for each $i < n+1$, $k_i < n$ and for each α in B_i passing through $\underline{0}p_i$, $\gamma|\alpha$ will belong to B_{k_i} . Without loss of generality we may assume $k_0 = k_1 = 0$. Let δ be a function from \mathcal{N} to \mathcal{N} such that, for every α , $(\delta|\alpha)^0 = \underline{0}p_0 * \alpha^0$ and $(\delta|\alpha)^1 = \underline{0}p_1 * \alpha^1$ and for every i such that $1 < i < n+1$, $(\delta|\alpha)^i = \underline{0}p_i * \underline{1}$.

Observe that α belongs to $D^2(A_1)$ if and only if $\delta|\alpha$ belongs to $D^2(A_1)$ if and only if $(\gamma|(\delta|\alpha))^0 = \underline{0}$. Therefore $D^2(A_1)$ reduces to A_1 . But, as we saw in Theorem 5.4, $D^2(A_1)$ does not reduce to A_1 . \square

5.7 Not surprisingly, the facts reported in the last few Theorems have their counterparts in the domain of the real numbers. We leave it to the reader to define an operation Perhaps for subsets of \mathbb{R} like we did for subsets of \mathcal{N} .

5.8 Theorem:

- (i) The union of the closed real intervals $[0, 1]$ and $[1, 2]$ does not really-coincide with the closed real interval $[0, 2]$.
 $[0, 2]$ is the least closed set containing both $[0, 1]$ and $[1, 2]$, and also the least perhapsive set containing both $[0, 1]$ and $[1, 2]$.
 $[0, 2]$ coincides with Perhaps($[0, 1] \cup [1, 2]$, $[0, 1] \cup [1, 2]$).
 $[0, 1] \cup [1, 2]$ is not perhapsive and does not belong to Π_2^0 .
- (ii) The open real interval $(0, 1)$ does not coincide with any finite union of closed sets.

Proof: We leave the proof of (i) to the reader.

The following argument for (ii) is due to the referee of an earlier version of this paper. Let n be a natural number and suppose X_0, X_1, \dots, X_{n-1} are really-closed sets such that $(0, 1)$ coincides with $X_0 \cup X_1 \cup \dots \cup X_{n-1}$. Let Y_0, Y_1, \dots, Y_{n-1} be really-open subsets of \mathbb{R} such that for each $i < n$, $X_i = \mathbb{R} \setminus Y_i$. Observe that 0 does not belong to any set Y_i , therefore,

$\forall i < n \neg \neg \exists m [(0, \frac{1}{m}) \text{ is a subset of } Y_i]$, and, by intuitionistic logic,
 $\neg \neg \forall i < n \exists m [(0, \frac{1}{m}) \text{ is a subset of } Y_i]$.

Observe that if $\forall i < n \exists m [(0, \frac{1}{m}) \text{ is a subset of } Y_i]$ we may find m such that $\forall i < n [(0, \frac{1}{m}) \text{ is a subset of } Y_i]$, and the number $\frac{1}{2m}$ will not belong to any X_i . Contradiction.

Therefore $\neg \forall i < n \exists m [(0, \frac{1}{m}) \text{ is a subset of } Y_i]$ and there do not exist closed sets X_0, \dots, X_{n-1} such that $(0, 1)$ coincides with $X_0 \cup X_1 \cup \dots \cup X_{n-1}$. \square

5.9 We return to Baire space \mathcal{N} . We give the new name T to the set CB_{2^*} that we introduced in Section 3.1, so T coincides with $\{0\} \cup \{\bar{0}n * \langle 1 \rangle * 0 | n \in \mathbb{N}\}$.

Observe that T contains all elements of Cantor space \mathcal{C} that assume the value 1 either not at all or exactly one time. Its closure \bar{T} consists of all elements of Cantor space \mathcal{C} that do not assume the value 1 two times.

The third statement of the following result improves upon conclusion (v) of Theorem 5.4.

5.10 Theorem:

- (i) For all closed sets X, Y , the set $\overline{X \cup Y}$ coincides with the set $(X \cup Y)^{\neg \neg}$.
- (ii) For each positive n , for every n -sequence X_0, \dots, X_{n-1} of closed sets, the set $\overline{X_0 \cup \dots \cup X_{n-1}}$ coincides with the set $(X_0 \cup \dots \cup X_{n-1})^{\neg \neg}$.
- (iii) For each positive n , the set T does not reduce to the set $D^n(A_1)$.
- (iv) For each positive n , the closure \bar{T} of T is a subset of $\bar{D}^n(A_1)$ but not of $D^n(A_1)$.

Proof: (i) Let X, Y be closed sets. Let C, D be decidable subsets of \mathbb{N} such that every α , α belongs to X if and only if, for each n , $\bar{\alpha}n$ belongs to C , and α belongs to Y if and only if, for each n , $\bar{\alpha}n$ belongs to D . Assume that α belongs to $\overline{X \cup Y}$, then for each n , either for every $m \leq n$, $\bar{\alpha}m$ belongs to C , or for every $m \leq n$, $\bar{\alpha}m$ belongs to D .

Observe that if there exists n such that $\bar{\alpha}n$ does not belong to C , then for every n , $\bar{\alpha}n$ belongs to D . Therefore, also if $\neg \neg$ (there exists n such that $\bar{\alpha}n$ does not belong to C), then for every n , $\bar{\alpha}n$ belongs to D , that is α belongs to Y .

So if $\alpha \notin X$, then $\alpha \in Y$, and consequently $\neg \neg (\alpha \in X \vee \alpha \in Y)$.

We thus see that $\overline{X \cup Y}$ forms part $(X \cup Y)^{\neg \neg}$.

It is obvious that $(X \cup Y)^{\neg \neg}$ is a subset of $\overline{X \cup Y}$.

(ii) The proof is left to the reader.

(iii) We use induction. We have seen, in Theorem 3.7, that the set $T = CB_{2^*}$ does not belong to Π_2^0 , so T is not closed and does not reduce to $D^1(A_1)$. Let n be a

natural number and assume that T does not reduce to $D^n(A_1)$. Suppose that γ is a function from \mathcal{N} to \mathcal{N} reducing T to $D^{n+1}(A_1)$. Calculate i such that $i < n + 1$ and $(\gamma|\underline{0})^i = \underline{0}$. Without endangering generality we assume $i = n$, that is $(\gamma|\underline{0})^n = \underline{0}$. Using the First Axiom of Countable Choice, determine α in \mathcal{N} such that, for each j , $\alpha(j) < n + 1$ and $(\gamma|(\underline{0}j * \langle 1 \rangle * \underline{0}))^{\alpha(j)} = \underline{0}$. We claim that it is possible to decide, for each j , if there exists $k > j$ such that $\alpha(k) = n$ or not. For suppose j is a natural number and let β be the element of \mathcal{C} such that, for each p , $\beta(p) = 1$ if and only if p is the least $k > j$ such that $\alpha(k) = n$. Observe that $(\gamma|\beta)^n = \underline{0}$, therefore $\gamma|\beta$ belongs to $D^{n+1}(A_1)$, and β belongs to T , so either $\beta = \underline{0}$ or $\beta \# \underline{0}$. In the first case there does not exist $k > j$ such that $\alpha(k) = n$, in the second case there does.

Assume now that j is a natural number and there is no $k > j$ such that $\alpha(k) = n$. Observe that the sequence $\underline{0}$ belongs to the closure of the set $\{\beta|\beta \in \mathcal{N}|\exists i < n[(\gamma|\beta)^i = \underline{0}]\}$. As the latter set is a finite union of closed sets, we use (ii) and conclude $\neg\neg\exists i < n[(\gamma|\underline{0})^i = \underline{0}]$. If there exist $i < n$ such that $(\gamma|\underline{0})^i = \underline{0}$, we consider a function δ from \mathcal{N} to \mathcal{N} such that, for every β, n the function δ maps the sequence $\underline{0}n * \langle 1 \rangle * \beta$ onto the sequence $\gamma|(\underline{0}(n + j) * \langle 1 \rangle * \beta)$ and, for every β , if $\beta(0)$ differs from both 0, 1, then $\delta|\beta = \gamma|\beta$. Observe that δ reduces T to $D^n(A_1)$. According to the induction hypothesis, T does not reduce to $D^n(A_1)$ so there is no $i < n$ such that $(\gamma|\underline{0})^i = \underline{0}$. We conclude that we can never choose the first of the above-mentioned two alternatives, therefore, for every j , there exists $k > j$ such that $\alpha(k) = n$. We now define a strictly increasing sequence ζ in \mathcal{N} such that for each n , $\alpha(\zeta(n)) = n$. We construct a function δ from \mathcal{N} to \mathcal{N} such that, for every β, n , the function δ maps the sequence $\underline{0}n * \langle 1 \rangle * \beta$ onto the sequence $\gamma|(\underline{0}\zeta(n) * \langle 1 \rangle * \beta)$, and for every β , if $\beta(0)$ differs from both 0, 1, then $\delta|\beta = \gamma|\beta$. Observe that, for every β , β belongs to T if and only if $(\delta|\beta)^n = \underline{0}$.

Therefore T is closed. Contradiction.

(iv) The proof is left to the reader. \square

5.11 It follows from the fifth statement of Theorem 4.5 that for every enumerable and discrete subset D of \mathcal{N} that is also dense-in-itself there are uncountably many sets X with the property $D \subseteq X \subseteq D^{\neg\neg}$. This is a consequence of the fact that such a set D has unbounded perhapsity. We now intend to show that there are also very many sets X with the property $T \subseteq X \subseteq T^{\neg\neg}$, although \overline{T} coincides with Perhaps(T, T) and T has perhapsity 1*. We need some preparations.

Let f be a function from Cantor space \mathcal{C} to itself such that for every α , for every n , $(f|\alpha)(n) = 1$ if and only if there exists $m \leq n$ such that $n = \overline{\alpha}m$. Observe that for all α in \mathcal{C} there exist infinitely many j such that $(f|\alpha)(j) = 1$ and that for all α, β in \mathcal{C} , if $\alpha \# \beta$, then there are only finitely many j such that $\alpha(j) = \beta(j) = 1$.

Let $\dot{-}$ be the binary operation on \mathcal{C} that is defined by:

for all α, β in \mathcal{C} , for all n , $(\alpha \dot{-} \beta)(n) := \alpha(n) - \beta(n)$.

Let Max be the binary operation on \mathcal{C} that is defined by: for all α, β in \mathcal{C} , for all n ,

$$(\text{Max}(\alpha, \beta))(n) = \text{Max}(\alpha(n), \beta(n)).$$

For every α in \mathcal{C} , we define subsets $\mathbb{T}(\alpha)$ and $\mathbb{U}(\alpha)$ of \mathcal{C} , as follows:

$$\mathbb{T}(\alpha) := \{\underline{0}\} \cup \{\underline{0}n * \langle 1 \rangle * \underline{0} \mid n \in \mathbb{N} \text{ and } \alpha(n) = 1\} \text{ and } \mathbb{U}(\alpha) := \overline{\mathbb{T}(f|\alpha)} \cup \mathbb{T}(\underline{1} \dot{-} (f|\alpha)).$$

Observe that, for every α in \mathcal{C} the closure $\overline{\mathbb{T}(\alpha)}$ of $\mathbb{T}(\alpha)$ is a spread containing $\underline{0}$.

5.12 Theorem:

- (i) For all α, β, γ in \mathcal{C} , if $\overline{\mathbb{T}(\alpha)}$ is a subset of $\overline{\mathbb{T}(\beta)} \cup \overline{\mathbb{T}(\gamma)}$, then either there exists n such that for every $j > n$, if $\alpha(j) = 1$, then $\beta(j) = 1$ or there exists n such that for every $j > n$, if $\alpha(j) = 1$ then $\gamma(j) = 1$.
- (ii) For all α, β in \mathcal{C} , $\overline{\mathbb{T}(\alpha)} \cup \overline{\mathbb{T}(\beta)} = \overline{\mathbb{T}(\text{Max}(\alpha, \beta))}$.
- (iii) For all α, β in \mathcal{C} , if $\mathbb{U}(\alpha)$ is a subset of $\mathbb{U}(\beta)$, then $\alpha = \beta$.
- (iv) For each α in \mathcal{C} , $T \subseteq \mathbb{U}(\alpha) \subseteq T^{\neg\neg}$, and the set $D^2(A_1)$ reduces to $\mathbb{U}(\alpha)$ and $\mathbb{U}(\alpha)$ does not coincide with either T or $T^{\neg\neg}$.
- (v) There is no set C such that $T \subseteq C \subseteq T^{\neg\neg}$ and the set $D^3(A_1)$ reduces to the set C .
- (vi) For each positive n there exists a set C such that $T \subseteq C \subseteq T^{\neg\neg}$ and C is a union of $n + 1$ closed sets not coinciding with any union of n closed sets.
- (vii) There is a set C such that $T \subseteq C \subseteq T^{\neg\neg}$ and C is a countable union of closed sets not coinciding with any finite union of closed sets.

Proof: (i) Suppose that $\overline{\mathbb{T}(\alpha)}$ is a subset of $\overline{\mathbb{T}(\beta)} \cup \overline{\mathbb{T}(\gamma)}$. Using the Continuity Principle we find n such that either every δ in $\overline{\mathbb{T}(\alpha)}$ passing through $\underline{0}n$ belongs to $\overline{\mathbb{T}(\beta)}$ or every δ in $\overline{\mathbb{T}(\alpha)}$ passing through $\underline{0}n$ belongs to $\overline{\mathbb{T}(\gamma)}$. If the first alternative obtains, then for all $j > n$, $\alpha(j) = 1$ entails $\beta(j) = 1$, and if the second one does, then for all $j > n$, $\alpha(j) = 1$ entails $\gamma(j) = 1$.

(ii) easily follows from (i).

(iii) Suppose that $\mathbb{U}(\alpha)$ is a subset of $\mathbb{U}(\beta)$. In particular, $\mathbb{T}(\underline{1} \dot{-} (f|\alpha))$ is a subset of $\overline{\mathbb{T}(f|\beta)} \cup \mathbb{T}(\underline{1} \dot{-} (f|\beta))$. Observe that there is no n such for every $j > n$, $(f|\alpha)(j) = 0$ entails $(f|\beta)(j) = 1$, therefore, by (i), there exists n such that for every $j > n$, $(f|\alpha)(j) = 0$ entails $(f|\beta)(j) = 0$, and thus $\alpha = \beta$.

(iv) Let α belong to \mathcal{C} . We may construct a function γ from \mathcal{N} to \mathcal{N} reducing $D^2(A_1)$ to $\mathbb{U}(\alpha) = \overline{\mathbb{T}(f|\alpha)} \cup \mathbb{T}(\underline{1} \dot{-} (f|\alpha))$. It suffices to ensure that for every β , for every n , *firstly*, if both β^0 and β^1 pass through $\underline{0}n$, then $\gamma|\beta$ passes through $\underline{0}n$, and *secondly*, if n is the least k such that exactly one of β^0, β^1 passes through $\underline{0}k$, then $\gamma|\beta$ must pass through some finite sequence $\underline{0}\ell * \langle 1 \rangle$, where $(f|\alpha)(\ell) = 1$ if β^0 passes through $\underline{0}n$ and $(f|\alpha)(\ell) = 0$ if β^1 passes through $\underline{0}n$, and *thirdly*, as long as at least one of β^0, β^1 passes through $\underline{0}n$, then $\overline{(\gamma|\beta)}$ assumes at most one time a value different from 0, but if neither one of β^0, β^1 passes through $\underline{0}n$, then $\gamma|\beta$ assumes two times a value different from 0 (and so does not belong to $T^{\neg\neg}$). If γ satisfies these conditions it will be clear that, for all β in \mathcal{C} , $\beta^0 = \underline{0}$ if and only if $\gamma|\beta$ belongs to $\overline{\mathbb{T}(f|\alpha)}$, and $\beta^1 = \underline{0}$ if and only if $\gamma|\beta$ belongs to $\mathbb{T}(\underline{1} \dot{-} (f|\alpha))$. So γ reduces $D^2(A_1)$ to $\mathbb{U}(\alpha)$.

Observe that $D^2(A_1)$ does not reduce to either T or $T^{\neg\neg}$, so $\mathbb{U}(\alpha)$ is different from both T and $T^{\neg\neg}$.

(v) Suppose C is a set such that $T \subseteq C \subseteq T^{\neg\neg}$ and γ is a function from \mathcal{N} to \mathcal{N} reducing $D^3(A_1)$ to C . We *claim* that γ maps every α in $(D^3(A_1))^{\neg\neg}$ onto $\underline{0}$. We prove this claim as follows:

Assume α belongs to $(D^3(A_1))^{\neg\neg}$ and $\gamma|\alpha \neq \underline{0}$. Let m be the least p such that $(\gamma|\alpha)(p) \neq 0$ and calculate n such that, for each β passing through $\overline{\alpha}n$, $(\gamma|\alpha)(m) = (\gamma|\beta)(m)$. Observe that we must have $(\gamma|\alpha)(m+1) = \underline{0}m * \langle 1 \rangle$, otherwise $\gamma|\alpha$ would not belong to T , and α itself would not belong to $D^3(A_1)$. There are two elements i of $\{0, 1, 2\}$ such that $\overline{\alpha^i}n = \underline{0}n$, and without loss of generality we may assume $\overline{\alpha^0}n = \overline{\alpha^1}n = \underline{0}n$. Define a function δ from \mathcal{N} to \mathcal{N} such that, for every β , $(\delta|\beta)^0 = \underline{0}n * \beta^0$, and $(\delta|\beta)^1 = \underline{0}n * \beta^1$, and $(\delta|\beta)n = \overline{\alpha}n$ and $(\delta|\beta)^2 \neq \underline{0}$. Observe that for every β , β belongs to $D^2(A_1)$ if and only if $(\gamma|(\delta|\beta))$ equals $\underline{0}m * \langle 1 \rangle * \underline{0}$, therefore $D^2(A_1)$ is closed. Contradiction. Therefore γ indeed maps $(D^3(A_1))^{\neg\neg}$ onto $\{\underline{0}\}$ and $(D^3(A_1))^{\neg\neg}$ is a subset of $D^3(A_1)$. Observe that $(D^3(A_1))^{\neg\neg}$ coincides with $\overline{D^3(A_1)}$ and $D^3(A_1)$ is not closed, see Theorems 5.4 and 5.10. Contradiction.

(vi) For each positive n , for each $i < n+1$, we let E_n^i be the closure of the set $\{\underline{0}\} \cup \{\underline{0}(k(n+1) + i) * \langle 1 \rangle * \underline{0} | k \in \mathbb{N}\}$, and we define $C_n := \bigcup_{i < n+1} E_n^i$. It will be clear that C_n is a union of $n+1$ closed sets and that $T \subseteq C_n \subseteq T^{\neg\neg}$. Suppose that we find closed sets F_0, F_1, \dots, F_{n-1} such that C_n coincides with $\bigcup_{i < n} F_i$. Applying the Continuity Principle $n+1$ times we find for each $i < n$ numbers m_i, n_i such that every α in E_n^i passing through $\underline{0}m_i$ belongs to F_{n_i} .

Without endangering generality we may assume $n_0 = n_1 = 0$. It easily follows that the closure of $E_n^0 \cup E_n^1$ forms part of $\bigcup_{i < n} F_i$ and therefore of C_n . Using once more the Continuity Principle, we obtain a contradiction.

(vii) For each i , we let E^i be the closure of the set $\{\underline{0}\} \cup \{\underline{0}(2^k \cdot (2i+1) - 1) * \langle 1 \rangle * \underline{0} | k \in \mathbb{N}\}$ and we define $C := \bigcup_{i \in \mathbb{N}} E^i$. It is not difficult to see that C satisfies the requirements. \square

5.13 For each positive n , there exists a subset of \mathbb{R} that is a union of $n+1$ closed sets and does not coincide with any union of n closed sets. In order to find such sets, one may start from the set T^* consisting of the real numbers really-coinciding with one of the rational numbers $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ and reconsider the sixth statement of Theorem 5.12.

5.14 Let X, Y be subsets of \mathcal{N} . We let *the conjunction of X and Y* , notation: $C(X, Y)$, be the set of all α such that α^0 belongs to X and α^1 belongs to Y .

More generally, let n be a positive natural number and let X_0, \dots, X_{n-1} be an n -sequence of subsets of \mathcal{N} . We let *the conjunction of X_0, \dots, X_{n-1}* , notation $C(X_0, \dots, X_{n-1})$ or $C_{i=0}^{n-1}(X_i)$, be the set of all α such that, for all $j < n$, α^j belongs to X_j .

Observe that for every positive n , for all subsets Z, X_0, \dots, X_{n-1} of \mathcal{N} , Z reduces to $C(X_0, \dots, X_{n-1})$ if and only if there exists an n -sequence Z_0, \dots, Z_{n-1} of subsets of \mathcal{N} such that $Z = Z_0 \cap \dots \cap Z_{n-1}$ and for each $j < n$, Z_j reduces to X_j .

Let X be a subset of \mathcal{N} and n a positive natural number. We let *the n -fold conjunction of X* , notation: $C^n(X)$, be the set of all α such that for all $j < n$, α^j belongs to X .

Observe that for every positive n , for all subsets Z, X of \mathcal{N} , Z reduces to $C^n(X)$ if and only if there exists an n -sequence Z_0, \dots, Z_{n-1} of subsets of \mathcal{N} , each of them reducing to X , such that $Z = Z_0 \cap \dots \cap Z_{n-1}$.

Recall that for every natural number m there exists a natural number

$k = \text{length}(m)$ and natural numbers $(m)_0, \dots, (m)_{k-1}$ such that

$$m = \langle (m)_0, \dots, (m)_{k-1} \rangle.$$

For all natural numbers m, n we define: m bows to n , notation: $m \prec n$, if and only if $\text{length}(m) = \text{length}(n)$ and for all $i < \text{length}(m)$, $(m)_i < (n)_i$. For all natural numbers m we let P_m be the set of all α such that for every $j < \text{length}(m)$, $\alpha^{j, (m)_j} = \underline{0}$.

Observe that, for every m , P_m is a spread containing $\underline{0}$.

For all natural numbers n , we let Q_n be the (finite) union of all sets P_m such that m bows to n . Observe that, for each n, k , if $k = \text{length}(n)$, then Q_n coincides with $C(D^{(n)_0}(A_1), \dots, D^{(n)_{k-1}}(A_1))$.

Observe that $Q_0 = Q_{\langle \rangle} = \mathcal{N}$ and, for each n , if there exists $i < \text{length}(n)$ such that $(n)_i = 0$, then $Q_n = \emptyset$.

For all n, j, p such that $j < \text{length}(n)$ and $p > 0$ we let $c = c(n, j, p)$ be the natural number satisfying the following conditions: $\text{length}(c) = \text{length}(n)$ and for all $i < \text{length}(n)$, if $i \neq j$, then $(c)_i = (n)_i$, and $(c)_j$ is the greatest natural number q such that $p \cdot q \leq (n)_j$.

5.15 Theorem:

- (i) For all positive natural numbers, p, q , the set $C(D^p(A_1), D^q(A_1))$ reduces to the set $D^{p \cdot q}(A_1)$.
- (ii) For all positive natural numbers p, m, n , the set $Q_{\langle p \rangle * m}$ reduces to the set Q_n if and only if there exists $j < \text{length}(n)$ such that the set Q_m reduces to the set $Q_{c(n, j, p)}$.

Proof: (i) Let p, q be positive natural numbers. Let γ be a function from \mathcal{N} to \mathcal{N} such that for all $i < p, j < q$, for all n , the number $(\gamma|\alpha)^{p \cdot j + i}(n)$ equals the number $\text{Min}(\alpha^{0, i}(n), \alpha^{1, j}(n))$. The function γ reduces the set $C(D^p(A_1), D^q(A_1))$ to the set $D^{p \cdot q}(A_1)$.

(ii) Let p, m, n be positive natural numbers and assume γ is a function from \mathcal{N} to \mathcal{N} reducing $Q_{\langle p \rangle * m}$ to Q_n . Using the Continuity Principle a finite number of times we find s in \mathbb{N} and a function F from the set of numbers bowing to $\langle p \rangle * m$ to the set of numbers bowing to n such that, for every $t \prec \langle p \rangle * m$, the function γ maps every α in P_t passing through $\bar{0}_s$ into the set $P_{F(t)}$.

We *claim* that there exists $j < \text{length}(n)$ such that for all t, u bowing to $\langle p \rangle * m$, if $(t)_0 \neq (u)_0$, then $(F(t))_j \neq (F(u))_j$.

For suppose there is no such j . Let X be the set of all α in Cantor space \mathcal{C} such that α^0 assumes at most one time the value 1 and for each i , if there is no q such that $i = \langle 0, q \rangle$, then $\alpha(i) = 0$. Observe that X is a spread containing $\underline{0}$. Remark also that for every α in X , for every $j < \text{length}(n)$, the sequence $\gamma|\alpha$ belongs to $D^{(n)_j}(A_1)$. Indeed, let j be a natural number and let t, u be numbers bowing to $\langle p \rangle * m$ such that $(t)_0 \neq (u)_0$ and $k := (F(t))_j = (F(u))_j$. Observe that for every α , if α belongs to $P_t \cup P_u$, then $(\gamma|\alpha)^{j,k} = \underline{0}$. As X forms part of $(P_t \cup P_u)^{\neg\neg}$, also for every α , if α belongs to X , then $(\gamma|\alpha)^{j,k} = \underline{0}$.

We conclude that γ maps X into Q_n , therefore X forms part of $Q_{\langle p \rangle * m}$, and, in particular, for every α in X there exists k such that $\alpha^{0,k} = \underline{0}$. Therefore the spread consisting of all α in \mathcal{C} that assume the value 1 at most one time forms part of $D^p(A_1)$. Using the Continuity Principle we find s, k such that every such α passing through $\bar{0}s$ has the property $\alpha^k = \underline{0}$. This is false, and our *claim* holds true.

Now choose $j < \text{length}(n)$ such that for all t, u bowing to $\langle p \rangle * m$, if $(t)_0 \neq (u)_0$, then $(F(t))_j \neq (F(u))_j$.

For each $k < p$ we let C_k be the set of all numbers $(F(t))_j$ where t is some number bowing to $\langle p \rangle * m$ such that $(t)_0 = k$.

Observe that for all $k, \ell < p$, if $k \neq \ell$, then C_k and C_ℓ are mutually disjoint subsets of the set $\{0, 1, \dots, (n)_j - 1\}$. We now determine $k < p$ such that for every $\ell < p$, the number of elements of C_k does not exceed the number of elements of C_ℓ . Observe that the number of elements of C_k does not exceed the greatest natural number q such that $p \cdot q \leq (n)_j$. In order to see that Q_m reduces to $Q_{c(n,j,p)}$ we define a function δ from \mathcal{N} to \mathcal{N} such that for every α , $(\delta|\alpha)^{0,k} = \underline{0}$ and for all $j < \text{length}(m)$, $(\delta|\alpha)^{j+1} = \alpha^j$. We then remark that for every α , α belongs to Q_m if and only if $\gamma|(\delta|\alpha)$ belongs to Q_n and, in addition, there is some i in C_k such that $(\gamma|(\delta|\alpha))^{j,i} = \underline{0}$. We conclude that, if $Q_{\langle p \rangle * m}$ reduces to Q_n , then there exists $j > n$ such that Q_m reduces to $Q_{c(n,j,p)}$.

Now assume that p, m, n are natural numbers such that for some $j < \text{length}(n)$, the set Q_m reduces to the set $Q_{c(n,j,p)}$. Using (i) one may prove that $Q_{\langle p \rangle * m}$ reduces to Q_n . \square

5.16 Theorem 5.15 enables us, given any m, n , to decide in finitely many steps if the set Q_m reduces to the set Q_n or not.

We may prove, for instance, that the sets $C^3(D^2(A_1))$ and $C^2(D^3(A_1))$ do not reduce to each other.

6 Forming limits and finding more hierarchies

We consider various upper bounds for a given sequence of subsets of \mathcal{N} . If upper bounds are taken repeatedly, hierarchies arise similar to the Cantor-Bendixson-

hierarchy discussed in Section 3.

6.1 For all subsets X, Y of \mathcal{N} we let *the (disjoint) sum of X and Y* , notation $X \oplus Y$, be the set $\langle 0 \rangle * X \cup \langle 1 \rangle * Y$.

For every sequence X_0, X_1, \dots of subsets of \mathcal{N} we let *the (countable) (disjoint) sum of the sequence X_0, X_1, \dots* , notation $\sum_{n \in \mathbb{N}} X_n$, be the set $\bigcup_{n \in \mathbb{N}} \langle n \rangle * X_n$.

The following Theorem establishes that (the reducibility degrees of) the subsets of \mathcal{N} behave as the elements of a countably complete upper semi-lattice.

6.2 Theorem:

- (i) For all subsets X, Y, Z of \mathcal{N} , the set $X \oplus Y$ reduces to Z if and only if both X and Y reduce to Z .
- (ii) For every sequence X_0, X_1, \dots of subsets of \mathcal{N} , for every subset Z of \mathcal{N} , the set $\sum_{n \in \mathbb{N}} X_n$ reduces to Z if and only if, for each n , X_n reduces to Z .

Proof: The proof is straightforward and left to the reader. One has to apply the Second Axiom of Countable Choice when proving (ii). \square

6.3 Let X_0, X_1, \dots be a sequence of subsets of \mathcal{N} . We consider a sample of four from the many sets Y there are with the property that each set X_n reduces to Y . We define:

$$\begin{aligned}
0 - \text{Lim}_{n \in \mathbb{N}}(X_n) &:= \bigcup_{n \in \mathbb{N}} \bar{0}n * \langle 1 \rangle * X_n, \\
1 - \text{Lim}_{n \in \mathbb{N}}(X_n) &:= \{\alpha \mid \alpha \in \mathcal{N} \mid \alpha = \bar{0} \text{ or } \alpha \text{ belongs to } 0 - \text{Lim}_{n \in \mathbb{N}}(X_n)\}, \\
2 - \text{Lim}_{n \in \mathbb{N}}(X_n) &:= \{\alpha \mid \alpha \in \mathcal{N} \mid \text{If } \alpha \# \bar{0}, \text{ then } \alpha \text{ belongs to } 0 - \text{Lim}_{n \in \mathbb{N}}(X_n)\}, \\
\text{and } 3 - \text{Lim}_{n \in \mathbb{N}}(X_n) &:= \bigcup_{n \in \mathbb{N}} \bar{0}n * \langle 1 \rangle * X_{2n} \cup \bigcup_{n \in \mathbb{N}} \bar{2}n * \langle 3 \rangle * X_{2n+1}
\end{aligned}$$

Suppose that, for each n , the set X_n coincides with the set $\{\bar{0}\}$. Observe that now, for all $i, j < 3$, if $i - \text{Lim}_{n \in \mathbb{N}}(X_n)$ reduces to $j - \text{Lim}_{n \in \mathbb{N}}(X_n)$, then $i = j$. Moreover, the set $1 - \text{Lim}_{n \in \mathbb{N}}(X_n)$ coincides with CB_{2*} and the set $2 - \text{Lim}_{n \in \mathbb{N}}(X_n)$ coincides with the closure $\overline{CB_{2*}}$ of CB_{2*} . Next assume that, for each n , the set X_{2n} coincides with $C^n(D^3(A_1))$ and the set X_{2n+1} coincides with $C(D^4(A_1), C^n(D^2(A_1)))$. Then the set $3 - \text{Lim}_{n \in \mathbb{N}}(X_n)$ reduces to the set $0 - \text{Lim}_{n \in \mathbb{N}}(X_n)$ but $0 - \text{Lim}_{n \in \mathbb{N}}(X_n)$ does not reduce to $3 - \text{Lim}_{n \in \mathbb{N}}(X_n)$.

We leave it to the reader to verify these statements.

It will be clear that every infinite sequence of subsets of \mathcal{N} admits of a wide variety of upper bounds.

6.4 Let n be a positive natural number and let X_0, \dots, X_{n-1} be an n -sequence of subsets of \mathcal{N} . We let *the disjunction of X_0, \dots, X_{n-1}* , notation $D(X_0, \dots, X_{n-1})$ or

$D_{i=0}^{n-1}(X_i)$, be the set of all α such that, for some $i < n$, α^i belongs to X_i .

We introduce, for every stump σ , a subset DCD_σ of \mathcal{N} , as follows, by transfinite induction:

- (i) $DCB_{\underline{1}} := \{\underline{0}\}$
- (ii) For every non-empty stump σ , $DCB_\sigma := \{\underline{0}\} \cup \bigcup_{n \in \mathbb{N}} \bar{0}n * \langle 1 \rangle * D_{i=0}^n(DCB_{\sigma^i})$.

We have added the letter D of Disjunction to the letters C, B of Cantor and Bendixson. We need a principle of induction in order to prove nice properties of this collection of sets. Some other variants of induction on the set of stumps are mentioned in [23].

Let $(\sigma_0, \dots, \sigma_{m-1})$ and $(\tau_0, \dots, \tau_{n-1})$ be finite sequences of stumps. We call the sequence $(\tau_0, \dots, \tau_{n-1})$ a *simplification* of the sequence $(\sigma_0, \dots, \sigma_{m-1})$ if and only if there exists $i < m$ and a positive natural number k such that σ_i is non-empty and $n = m + (k - 1)$ and for each $j < i$, $\sigma_j = \tau_j$, and for each $j < k$, $\tau_{i+j} = (\sigma_i)^j$ and, for each $j < m - i - 1$, $\tau_{i+k+j} = \sigma_{i+j+1}$. So the sequence $(\tau_0, \dots, \tau_{n-1})$ results from the sequence $(\sigma_0, \dots, \sigma_{m-1})$ if one replaces (σ_i) by $((\sigma_i)^0, \dots, (\sigma_i)^{k-1})$.

We denote the set of finite sequences of stumps by \mathbf{Stp}^* .

6.5 Theorem: (A principle of Induction on \mathbf{Stp}^*)

Let P be a subset of \mathbf{Stp}^* such that every finite sequence of stumps belongs to \mathbf{Stp}^* as soon as each one of its simplifications belongs to \mathbf{Stp}^* .

Then every finite sequence of stumps belongs to \mathbf{Stp}^* .

6.6 The proof of Theorem 6.5 requires some preparations and will be given only in Section 6.11.

Let σ be a stump. We let $\mathbb{B}(\sigma)$ be the set of all natural numbers belonging to σ , that is such that $\sigma(n) = 0$, see Section 1.5.3.

It is wise to think of $\mathbb{B}(\sigma)$ as a set of (code numbers of) finite sequences of natural numbers.

Let A be a countable set and $<_0$ be a binary relation on A . Let P be a subset of A . P is called *$<_0$ -hereditary* if and only if, for every a in A , a belongs to P as soon as every b in A such that $b <_0 a$ belongs to P . $<_0$ is called an *inductive* relation on A if and only if every $<_0$ -hereditary subset of A coincides with A . $<_0$ is called a *stumpy* relation on A if and only if there exists a stump σ such that $(A, <_0)$ embeds isomorphically into $(\mathbb{B}(\sigma), <^\#)$ where, for all m, n in \mathbb{N} , $m <^\# n$ if and only if there exists p such that $m = n * \langle p \rangle$, that is, m is, as a finite sequence, an immediate successor of n . So $(A, <_0)$ is stumpy if and only if there exists an injective mapping f from A into some $\mathbb{B}(\sigma)$ such that for every a_0, a_1 in A , $a_0 <_0 a_1$ if and only if $f(a_0) <^\# f(a_1)$.

Let A, B be countable sets and $<_0, <_1$ binary relation on A, B , respectively. We define a relation $<_2$ on the set $A \times B$ as follows. For all a_0, a_1 in A , b_0, b_1 in B , $(a_0, b_0) <_2 (a_1, b_1)$ if and only if either $a_0 <_0 a_1$ and $b_0 = b_1$ or $a_0 = a_1$ and

$b_0 < b_1$. We call the relation $<_2$ the *interweaving* of the relations $<_0, <_1$ and denote the structure $\langle A \times B, <_2 \rangle$ by $(A, <_0) \otimes (B, <_1)$.

6.7 Lemma: (*Principle of Double Induction on the set **Stp** of stumps*)

Let P be a subset of **Stp** \times **Stp** such that every pair $(\sigma, \underline{1})$ and every pair $(\underline{1}, \tau)$ belong to P , and every pair (σ, τ) of non-empty stumps belongs to P as soon as every pair (σ^n, τ) and every pair (σ, τ^n) belong to P .
Then every pair of stumps belongs to P .

Proof: Let Q be the set of all stumps σ that for every stump τ , the pair (σ, τ) belongs to P . Now use 1.5.2, the First Principle of Induction on the set **Stp** of stumps. \square

6.8 Lemma:

- (i) Let A be a countable set and $<_0$ a binary relation on A .
If A is stumpy, then $<_0$ is inductive.
- (ii) For all stumps σ, τ , the interweaving $(\mathbb{B}(\sigma), <^\#) \otimes (\mathbb{B}(\tau), <^\#)$ is stumpy.

Proof: (i) Let σ be a stump and let f be an isomorphic embedding from $(A, <_0)$ into $(\mathbb{B}(\sigma), <^\#)$. Let P be a $<_0$ -hereditary subset of Q . Let Q be the set of all m in $\mathbb{B}(\sigma)$ such that, for every a in A , if $f(a) = m$, then a belongs to P . Observe that Q is a $<^\#$ -hereditary subset of $\mathbb{B}(\sigma)$, therefore, by the principle of Stump Induction, Theorem 1.5.4, Q coincides with $\mathbb{B}(\sigma)$ and P coincides with A .

(ii) We use the just-mentioned Principle of Double Induction on the set of stumps, 6.7. Observe that the statement holds if either σ or τ is the empty stump. So assume that both σ and τ are non-empty and that for each n , there exist stumps $\rho_{n,0}$ and $\rho_{n,1}$ such that the interweaving $(\mathbb{B}(\sigma), <^\#) \otimes (\mathbb{B}(\tau^n), <^\#)$ embeds into $(\mathbb{B}(\rho_{n,0}), <^\#)$ and the interweaving $(\mathbb{B}(\sigma^n), <^\#) \otimes (\mathbb{B}(\tau), <^\#)$ embeds into $(\mathbb{B}(\rho_{n,1}), <^\#)$. Now form a stump φ such that for each n , $\varphi^{2n} = \rho_{n,0}$ and $\varphi^{2n+1} = \rho_{n,1}$ and observe that the interweaving $(\mathbb{B}(\sigma), <^\#) \otimes (\mathbb{B}(\tau), <^\#)$ embeds into $(\mathbb{B}(\varphi), <^\#)$. \square

6.9 Let A be a set and $<_0$ a binary relation on A .

We define a binary relation $<_1$ on the set A^* of finite sequences of elements of A . For all $(a_0, \dots, a_{m-1}), (b_0, \dots, b_{n-1})$ in A^* we define: $(b_0, \dots, b_{n-1}) (<_0)^s (a_0, \dots, a_{m-1})$, or: (b_0, \dots, b_{n-1}) is a $<_0$ -*simplification* of (a_0, \dots, a_{m-1}) , if and only if there exist $i < m$ and a positive natural number k such that $n = m + (k - 1)$, and for each $j < i$, $a_j = b_j$, and for each $j < k$, $b_{i+j} <_0 a_i$, and for each $j < m - i - 1$, $b_{i+k+j} = a_{i+j+1}$. So the sequence (b_0, \dots, b_{n-1}) is obtained from the sequence (a_0, \dots, a_{m-1}) if one replaces (a_i) by (b_i, \dots, b_{i+k-1}) where, for each $j < k$, $b_{i+j} <_0 a_i$.

We call the relation $(<_0)^s$ the *sequencing* of the relation $<_0$.

6.10 Lemma:

For every stump σ , the sequencing $(<^\#)^s$ of the relation $<^\#$ is a stumpy relation on $(\mathbb{B}(\sigma))^*$.

Proof: We use induction on the set of stumps. The statement of the Lemma is trivially true if σ is the empty stump. Now assume that σ is non-empty and that, for each n , the relation $(<^\#)^s$ is stumpy on $(\mathbb{B}(\sigma^n))^*$.

For all elements $a = (a_0, \dots, a_{m-1})$ and $b = (b_0, \dots, b_{n-1})$ of $(\mathbb{B}(\sigma))^*$ we define: $b <^* a$ if and only if there exists a finite sequence c_0, \dots, c_{k-1} of elements of $(\mathbb{B}(\sigma))^*$ such that $b = c_0$ and $a = c_{k-1}$ and, for each $j < k$, c_j is a $<^\#$ -simplification of c_{j+1} . So the relation $<^*$ is the transitive closure of the relation $(<^\#)^s$.

For each a in $(\mathbb{B}(\sigma))^*$ we let $(\mathbb{B}(\sigma))^* \upharpoonright a$ be the set of all b in $(\mathbb{B}(\sigma))^*$ such that $b <^* a$.

Observe that it suffices to show that, for each non-empty sequence a in $(\mathbb{B}(\sigma))^*$, the structure $(\mathbb{B}(\sigma)^* \upharpoonright a, (<^\#)^s)$ is stumpy.

We prove this by induction on $\text{length}(a)$.

If a has length 1, we determine t in \mathbb{N} such that $a = \langle t \rangle$. We may assume that t is positive and determine $k > 0$ such that $t = \langle t(0), \dots, t(k-1) \rangle$.

We observe that $a <^* \langle t(0) \rangle$ and use the fact that $(<^\#)^s$ is stumpy on $(\mathbb{B}(\sigma^{t(0)}))^*$.

If a has length greater than 1, we determine b in $(\mathbb{B}(\sigma))^*$ and t in \mathbb{N} such that $a = b * \langle t \rangle$.

We may assume that t is coding a non-empty finite sequence $t = \langle t(0), \dots, t(k-1) \rangle$.

We also may assume that $(<^\#)^s$ is stumpy on $(\mathbb{B}(\sigma))^* \upharpoonright b$. By the induction hypothesis $(<^\#)^s$ is stumpy on $(\mathbb{B}(\sigma))^* \upharpoonright \langle t \rangle$ as $(\mathbb{B}(\sigma))^* \upharpoonright \langle t \rangle$ forms part of $(\mathbb{B}(\sigma^{t(0)}))^*$.

Now observe that $((\mathbb{B}(\sigma))^* \upharpoonright a, (<^\#)^s)$ may be seen as the result of interweaving $((\mathbb{B}(\sigma))^* \upharpoonright b, (<^\#)^s)$ and $((\mathbb{B}(\sigma))^* \upharpoonright \langle t \rangle, (<^\#)^s)$ and conclude by Lemma 6.9 that $(<^\#)^s$ is stumpy on $(\mathbb{B}(\sigma))^* \upharpoonright a$. \square

6.11 Proof of Theorem 6.5.

Let P be a subset of \mathbf{Stp}^* such that every finite sequence of stumps belongs to P as soon as each one of its simplifications belongs to P .

Let $(\sigma_0, \dots, \sigma_{n-1})$ be a finite sequence of stumps. Let τ be some stump such that for each $i < n$, $\tau^i = \sigma_i$. For each t in $\mathbb{B}(\tau)$ we define a stump ${}^t\tau$ as follows. The definition is by induction on $\text{length}(t)$. We define ${}^0\tau := \tau$ and for each t, i , if $t * \langle i \rangle$ belongs to $\mathbb{B}(\tau)$, then ${}^{t * \langle i \rangle}\tau = ({}^t\tau)^i$.

Observe that for all elements $a = (a(0), \dots, a(m-1))$ and $b = (b_0, \dots, b_{n-1})$ of $\mathbb{B}(\tau)^*$, (b_0, \dots, b_{n-1}) is a $<^\#$ -simplification of (a_0, \dots, a_{m-1}) if and only if the sequence $\langle {}^{b(0)}\tau, \dots, {}^{b(n-1)}\tau \rangle$ is a simplification of the sequence $\langle {}^{a(0)}\tau, \dots, {}^{a(m-1)}\tau \rangle$. Now let Q be the set of all elements $(a(0), \dots, a(m-1))$ of $(\mathbb{B}(\tau))^*$ such that $({}^{a(0)}\tau, \dots, {}^{a(m-1)}\tau)$ belongs to P . Conclude by Lemma 6.10 that Q coincides with $(\mathbb{B}(\tau))^*$, in particular $(\langle 0 \rangle, \langle 1 \rangle, \dots, \langle n-1 \rangle)$ belongs to Q , therefore $(\langle 0 \rangle\tau, \dots, \langle n-1 \rangle\tau)$ belongs to P , that is $(\sigma_0, \dots, \sigma_{n-1})$ belongs to P . \square

6.12 Theorem:

- (i) For each stump σ , the set DCB_σ belongs to the class Σ_2^0 , and its closure $\overline{DCB_\sigma}$ coincides with its double complement $(DCB_\sigma)^{\neg\neg}$.
- (ii) For all hereditarily repetitive stumps σ, τ , if $\sigma \leq \tau$ then the set DCB_σ reduces to the set DCB_τ .
- (iii) For every stump σ , for every n , the set DCB_σ reduces to the set $DCB_\sigma \upharpoonright \bar{0}n$.
- (iv) For every finite sequence $(\sigma_0, \sigma_1, \dots, \sigma_{n-1})$ of stumps, the set $D(A_1, D_{i=0}^{n-1}(DCB_{\sigma_i}))$ does not reduce to the set $D_{i=0}^{n-1}(DCB_{\sigma_i})$.
- (v) For all hereditarily repetitive stumps σ, τ , if $\sigma < \tau$, then the set DCB_τ does not reduce to the set DCB_σ .
- (vi) For each stump σ , for each n , the set $D^{n+1}(DCB_\sigma)$ does not reduce to the set $D^n(DCB_\sigma)$.

Proof: (i) We leave the proof to the reader as it is similar to the proof of Theorem 3.5(ii).

(ii) We use induction on the set of hereditarily repetitive stumps. It is obvious that for each stump τ , the set $DCB_{\bar{1}}$ reduces to the set DCB_τ . Now assume that σ, τ are hereditarily repetitive stumps, σ is non-empty and $\sigma \leq \tau$. Using our Axioms of Countable Choice we find a strictly increasing α such that, for each m , $\sigma^m \leq \tau^{\alpha(m)}$ and also γ such that, for each m , γ^m is a function from \mathcal{N} to \mathcal{N} reducing the set DCB_{σ^m} to the set $DCB_{\tau^{\alpha(m)}}$.

We leave it to the reader to define δ in such a way that, for each m , δ^m is a function from \mathcal{N} to \mathcal{N} reducing the set $D_{i=0}^m(DCB_{\sigma^i})$ to the set $D_{i=0}^{\alpha(m)}(DCB_{\tau^i})$. Finally we construct a function ζ from \mathcal{N} to \mathcal{N} such that for every m , for every ε , the sequence $\zeta(\bar{0}m * \langle 1 \rangle * \varepsilon)$ equals the sequence $\bar{0}\alpha(m) * \langle 1 \rangle * (\delta^m \upharpoonright \varepsilon)$.

It will be clear that ζ maps $\bar{0}$ onto $\bar{0}$ and reduces DCB_σ to DCB_τ .

(iii) We leave the proof to the reader observing only that for each stump σ , for each n , the set $D_{i=0}^n(DCB_{\sigma^i})$ reduces to the set $D_{i=0}^{n+1}(DCB_{\sigma^i})$.

(iv) We intend to use the Principle of Induction on the set \mathbf{Stp}^* of finite sequences of stumps expressed in Theorem 6.5. Let $(\sigma_0, \sigma_1, \dots, \sigma_{n-1})$ be a finite sequence of stumps and assume that the statement has been proved for every finite sequence of stumps that is a simplification of the sequence $(\sigma_0, \sigma_1, \dots, \sigma_{n-1})$. (Observe that, if the sequence $(\sigma_0, \sigma_1, \dots, \sigma_{n-1})$ has no simplifications, then the statement to be proved is equivalent to the statement: $D^{n+1}(A_1)$ does not reduce the set $D^n(A_1)$ and therefore true by Theorem 5.6(iv). But the argument we are about to explain furnishes another proof of this special case). Let us assume that γ is a function from \mathcal{N} to \mathcal{N} reducing the set $D(A_1, D_{i=0}^{n-1}(DCB_{\sigma_i}))$ to the set $D_{i=0}^{n-1}(DCB_{\sigma_i})$. Consider the sets B_0, B_1, \dots, B_n which are defined as follows: $B_0 := \{\alpha \mid \alpha \in \mathcal{N} \mid \alpha^0 = \bar{0}\}$ and, for each $i < n$, $B_{i+1} := \{\alpha \mid \alpha \in \mathcal{N} \mid \alpha^{1,i} = \bar{0}\}$. Observe that every one of these sets is a spread containing $\bar{0}$ and forming part of $D(A_1, D_{i=0}^{n-1}(DCB_{\sigma_i}))$. Applying the Continuity Principle $n+1$ times we find for each $i \leq n$ natural numbers m_i and k_i such that, for

every α in B_i passing through $\bar{0}m_i$, the sequence $(\gamma|\alpha)^i$ belongs to $DCB_{\sigma_{k_i}}$. Without loss of generality we assume $k_0 = k_1 = 0$.

Applying the Continuity Principle two more times we find p_0, p_1 , such that for every $i < 2$ either $t_i = 0$ and for every α in B_i passing through $\bar{0}p_i$, the sequence $(\gamma|\alpha)^0$ coincides with $\bar{0}$, or $t_i = 1$ and for every α in B_i passing through $\bar{0}p_i$, there exists β such that $(\gamma|\alpha)^0$ equals $\bar{0}n_i * \langle 1 \rangle * \beta$ and β belongs to $D_{i=0}^{n_i}(DCB_{(\sigma_0)^i})$. Let us first assume $t_0 = t_1 = 0$. Observe that now for every α in B_0 passing through $\bar{0}p_0$ the sequence $(\gamma|\alpha)^0$ coincides with $\bar{0}$, and also for every α in B_1 passing through $\bar{0}p_1$, the sequence $(\gamma|\alpha)^0$ coincides with $\bar{0}$. We now see that the set $D^2(A_1)$ reduces to the set A_1 , as follows. We construct a function δ from \mathcal{N} to \mathcal{N} such that, for every α , $(\delta|\alpha)^0 = \bar{0}p_0 * \alpha^0$ and if $\alpha^1 = \bar{0}$, then $(\delta|\alpha)^{1,0} = \bar{0}$, but if $\alpha^1 \neq \bar{0}$, then $(\delta|\alpha)^{1,0}$ does not belong to DCB_{σ_0} , and for each i , if $0 < i \leq n-1$, then $(\delta|\alpha)^{1,i}$ does not belong to DCB_{σ_i} , and $(\delta|\alpha)p = \bar{0}p$ where $p = \max(p_0, p_1)$. It is not difficult to verify that for each α , α belongs to $D^2(A_1)$ if and only if $\delta|\alpha$ belongs to $D(A_1, D_{i=1}^{n-1}(DCB_{\sigma_i}))$ if and only if $(\gamma|(\delta|\alpha))^0 = \bar{0}$. We know, however, from Theorem 5.4(i) that the set $D^2(A_1)$ is not closed. So we conclude that either t_0 or t_1 differs from 0, and, without loss of generality we may assume that $t_0 = 1$. We now calculate q such that for every α , if α passes through $\bar{0}q$, then $(\gamma|\alpha)^0$ passes through $\bar{0}n_0 * \langle 1 \rangle$. Let ε be a function from \mathcal{N} to \mathcal{N} such that, for every α , the sequence $(\gamma|(\bar{0}q * \alpha))^0$ equals $\bar{0}n_0 * \langle 1 \rangle * (\varepsilon|\alpha)$. Observe that, for every α , $\bar{0}q * \alpha$ belongs to $D(A_1, D_{i=0}^{n-1}(DCB_{\sigma_i}))$ if and only if either $\varepsilon|\alpha$ belongs to $D_{i=0}^{n_0}(DCB_{(\sigma_0)^i})$ or for some positive $i < n$, the sequence $(\gamma|\bar{0}q * \alpha)^i$ belongs to DCB_{σ_i} . Using (iii) we conclude that $D(A_1, D_{i=0}^{n-1}(DCB_{\sigma_i}))$ reduces to $D(D_{i=0}^{n_0}(DCB_{(\sigma_0)^i}), D_{i=0}^{n-2}(DCB_{\sigma_{i+1}}))$. Observe that $D_{i=0}^{n_0}(DCB_{(\sigma_0)^i})$ reduces to DCB_{σ_0} , and therefore $D(A_1, D(D_{i=0}^{n_0}(DCB_{(\sigma_0)^i}), D_{i=0}^{n-2}(DCB_{\sigma_{i+1}})))$ reduces to $D(A_1, D_{i=0}^{n-1}(DCB_{\sigma_i}))$ and therefore also to $D(D_{i=0}^{n_0}(DCB_{(\sigma_0)^i}), D_{i=0}^{n-2}(DCB_{\sigma_{i+1}}))$. We now obtain a contradiction by our induction hypothesis, as the finite sequence $\langle (\sigma_0)^0, \dots, (\sigma_0)^{n_0-1}, \sigma_1, \dots, \sigma_{n-1} \rangle$ is a simplification of the finite sequence $\langle \sigma_0, \sigma_1, \dots, \sigma_{n-1} \rangle$.

(v) This easily follows from (iv). Let σ, τ be hereditarily repetitive stumps such that $\sigma < \tau$. Calculate m such that $\sigma \leq \tau^m$. Observe that DCB_σ reduces to DCB_{τ^m} . On the other hand $D_{i=0}^{m+1}(DCB_{\tau^i})$ reduces to DCB_τ but not to DCB_{τ^m} . Therefore DCB_τ does not reduce to DCB_σ .

(vi) This follows immediately from (iv). \square

6.13 We introduce, for every stump σ , a subset CCB_σ of \mathcal{N} , as follows, by transfinite induction:

- (i) $CCB_{\bar{1}} := D^2(A_1)$
- (ii) For every non-empty stump σ , $CCB_\sigma := \{\bar{0}\} \cup \bigcup_{n \in \mathbb{N}} \bar{0}n * \langle 1 \rangle * C_{i=0}^n(CCB_{\sigma^i})$.

We have added the letter C of Conjunction to the letters C, B of Cantor and Bendixson.

6.14 Theorem:

- (i) For each stump σ , the set CCB_σ , belongs to the class Σ_2^0 and its closure $\overline{CCB_\sigma}$ coincides with its double complement $(CCB_\sigma)^{\neg\neg}$.
- (ii) For all hereditarily repetitive stumps σ, τ , if $\sigma \leq \tau$, then the set CCB_σ reduces to the set CCB_τ .
- (iii) For every stump σ , for every n , the set CCB_σ reduces to the set $CCB_\sigma \upharpoonright \bar{0}n$.
- (iv) For every finite sequence $(\sigma_0, \sigma_1, \dots, \sigma_{n-1})$ of stumps, the set $C(D^2(A_1), C_{i=0}^{n-1}(CCB_{\sigma_i}))$ does not reduce to the set $C_{i=0}^{n-1}(CCB_{\sigma_i})$.
- (v) For all hereditarily repetitive stumps σ, τ , if $\sigma < \tau$, then the set CCB_τ does not reduce to the set CCB_σ .
- (vi) For each stump σ , for each n , the set $C^{n+1}(CCB_\sigma)$ does not reduce to the set $C^n(CCB_\sigma)$.

Proof: We only prove (iv) and leave it to the reader to prove the remaining statements of the Theorem.

We again want to use the Principle of Induction on the set of finite sequences of stumps expressed in Theorem 6.5. For every finite sequence $(\sigma_0, \dots, \sigma_{n-1})$ of stumps we define: $(\sigma_0, \dots, \sigma_{n-1})$ has the property $P :=$ either there exists $i < n$ such that $\sigma_i = \underline{1}$ or for every k , the set $C(C^{k+1}(D^2(A_1)), C_{i=0}^{n-1}(CCB_{\sigma_i}))$ does not reduce to the set $C(C^k(D^2(A_1)), C_{i=0}^{n-1}(CCB_{\sigma_i}))$.

It is easy to see that it suffices to show that every finite sequence of stumps has the property P .

Let us assume that $(\sigma_0, \dots, \sigma_{n-1})$ is a finite sequence of stumps such that every simplification of the finite sequence $(\sigma_0, \dots, \sigma_{n-1})$ has the property P . We may assume that none of the stumps $\sigma_0, \dots, \sigma_{n-1}$ is the empty stump. Suppose we find k in \mathbb{N} and a function γ from \mathcal{N} to \mathcal{N} reducing the set $C(C^{k+1}(D^2(A_1)), C_{i=0}^{n-1}(CCB_{\sigma_i}))$ to the set $C(C^k(D^2(A_1)), C_{i=0}^{n-1}(CCB_{\sigma_i}))$. For every $k+1$ -sequence $i = (i_0, \dots, i_k)$ from $\{0, 1\}^{k+1}$ we let $B(i_0, \dots, i_k)$ be the set of all α in \mathcal{N} such that, for every $j \leq k$, the sequence α^{0,j,i_j} coincides with $\underline{0}$ and, for every $j \leq n-1$, the sequence $\alpha^{1,j}$ coincides with $\underline{0}$. Observe that each set $B(i_0, \dots, i_k)$ is a spread containing $\underline{0}$ and forming part of $C(C^{k+1}(D^2(A_1)), C_{i=0}^{n-1}(CCB_{\sigma_i}))$.

Repeatedly applying the Continuity Principle we find, for each finite sequence (i_0, \dots, i_k) from $\{0, 1\}^*$, for each $j \leq n-1$, natural numbers $n = n(i_0, \dots, i_k, j)$ and $t = t(i_0, \dots, i_k, j)$ such that *either* $t = 0$ and for every α in $B(i_0, \dots, i_k)$ passing through $\bar{0}n$, the sequence $(\gamma|\alpha)^{1,j}$ coincides with $\underline{0}$, *or* $t = 1$ and for every α in $B(i_0, \dots, i_k)$, the sequence $(\gamma|\alpha)^{1,j}$ is apart from $\underline{0}$.

We claim that there must exist a finite sequence (i_0, \dots, i_k) in $\{0, 1\}^*$ and $j \leq n-1$ such that $t = t(i_0, \dots, i_k, j) = 1$. For suppose not. We then calculate $N = \max\{n(i_0, \dots, i_k, j) | (i_0, \dots, i_k) \in \{0, 1\}^{k+1}, j \leq n-1\}$ and construct a function δ from \mathcal{N} to \mathcal{N} such that for every $j < k+1$, the sequence $(\delta|\alpha)^{0,j}$ coincides with $\bar{0}N * \alpha^j$.

and for every $j \leq n-1$, the sequence $(\delta|\alpha)^{1,j}$ coincides with $\underline{0}$, and $(\delta|\alpha)(0) = 0$. Observe that, for every α , α belongs to $C^{k+1}(D^2(A_1))$ if and only if the sequence $(\gamma(\delta|\alpha))^0$ belongs to $C^k(D^2(A_1))$. Therefore the set $C^{k+1}(D^2(A_1))$ reduces to the set $C^k(D^2(A_1))$, and we have a contradiction, according to Theorem 5.15.

Without loss of generality we may assume that $t = t(0, \dots, 0) = 1$. We now determine p, q such that for every α in $B(0, \dots, 0)$ passing through $\underline{0}p$ the sequence $(\gamma|\alpha)^{1,0}$ passes through $\underline{0}q * \langle 1 \rangle$. As in the proof of Theorem 6.12 we may conclude that the set

$C(C^{k+1}(D^2(A_1)), C_{i=0}^{n-1}(CCB_{\sigma_i}))$ reduces to the set $C(C^k(D^2(A_1)), C(C_{i=0}^q(CCB_{(\sigma_0)^i}), C_{i=0}^{n-2}(CCB_{\sigma_{i+1}})))$.

But then also the set $C((C^{k+1}(D^2(A_1)), C(C_{i=0}^q(CCB_{(\sigma_0)^i}), C_{i=0}^{n-2}(CCB_{\sigma_{i+1}})))$ will reduce to the set $C(C^k(D^2(A_1)), C(C_{i=0}^q(CCB_{(\sigma_0)^i}), C_{i=0}^{n-2}(CCB_{\sigma_{i+1}})))$, and we obtain a contradiction, as the finite sequence

$((\sigma_0)^0, \dots, (\sigma_0)^q, \sigma_1, \dots, \sigma_{n-1})$ is a simplification of the finite sequence $(\sigma_0, \dots, \sigma_{n-1})$. □

7 The Borel Hierarchy Theorem

We introduce positively Borel sets and canonical classes of positively Borel sets. We discuss the custom of calling some given positively Borel sets X, Y each other's complement and remark that almost every positively Borel set has very many complements. We show which conclusion one may draw from the argument given by Borel and Lebesgue.

We then prove the Hierarchy Theorem, first for the finite levels only, and then, reasoning more shrewdly, the general case.

7.1 The class **Borel** of *positively Borel subsets* of \mathcal{N} is given by the following inductive definition.

- (i) Every subset of \mathcal{N} belonging to either $\mathbf{\Pi}_1^0$ or $\mathbf{\Sigma}_1^0$ is positively Borel.
- (ii) For any given sequence X_0, X_1, \dots of positively Borel subsets of \mathcal{N} , the sets $\bigcap_{n \in \mathbb{N}} X_n$ and $\bigcup_{n \in \mathbb{N}} X_n$ are themselves positively Borel.
- (iii) Clauses (i) and (ii) produce all positively Borel subsets of \mathcal{N} .

7.2 We define the class of the *non-zero stumps* by the following inductive definition: a stump σ is non-zero if either σ coincides with $\{\langle \rangle\} = 1^*$ or σ is non-empty and for each n , σ^n is a non-zero stump.

Every non-zero stump is non-empty but the converse is false.

Observe that we may decide, for every non-zero stump σ , if σ equals 1^* or not.

For every non-zero stump σ we define classes $\mathbf{\Sigma}_\sigma^0$ and $\mathbf{\Pi}_\sigma^0$ of subsets of \mathcal{N} , by the following inductive definition:

- (i) $\Sigma_{1^*}^0$ coincides with the class Σ_1^0 of the open subsets of \mathcal{N} and $\Pi_{1^*}^0$ coincides with the class Π_1^0 of the closed subsets of \mathcal{N} .
- (ii) For every non-zero stump σ different from 1^* , for every subset X of \mathcal{N} :
 X belongs to Σ_σ^0 if and only if there exist a sequence X_0, X_1, \dots of subsets of \mathcal{N} , and a sequence p_0, p_1, \dots of natural numbers such that for each n , X_n belongs to $\Pi_{\sigma^{p_n}}^0$ and X coincides with $\bigcup_{n \in \mathbb{N}} X_n$, and:
 X belongs to Π_σ^0 if and only if there exist a sequence X_0, X_1, \dots of subsets of \mathcal{N} and a sequence p_0, p_1, \dots of natural numbers such that, for each n , X_n belongs to $\Sigma_{\sigma^{p_n}}^0$ and X coincides with $\bigcap_{n \in \mathbb{N}} X_n$.

The classes $\Sigma_\sigma^0, \Pi_\sigma^0$ are called *the canonical classes of positively Borel sets*.

Observe that a subset X of \mathcal{N} is positively Borel if and only if, for some non-zero stump σ , X belongs to Σ_σ^0 .

7.3 We define the class of *complementary pairs (of positively Borel sets)* by the following definition.

- (i) For every open subset X of \mathcal{N} , the two-element collection $\{X, \mathcal{N} \setminus X\}$ is a complementary pair.
- (ii) For every sequence $X_0, Y_0, X_1, Y_1, \dots$ of positively Borel sets, if, for each n , $\{X_n, Y_n\}$ is a complementary pair, then the two-element collection $\{\bigcup_{n \in \mathbb{N}} X_n, \bigcap_{n \in \mathbb{N}} Y_n\}$ is a complementary pair.
- (iii) Clauses (i) and (ii) produce all complementary pairs.

7.3.1 Theorem:

- (i) For every positively Borel set X there exists a positively Borel set Y such that $\{X, Y\}$ is a complementary pair.
- (ii) For every complementary pair $\{X, Y\}$, every element of X is apart from every element of Y .
- (iii) **Inf** is the set of all elements in \mathcal{C} apart from every element of **Fin**. **Almost*(Fin)** is the set of all elements of \mathcal{C} apart from every element of **Inf**. For every stump σ , the two-element collection $\{\mathbb{P}(\sigma, \mathbf{Fin}), \mathbf{Inf}\}$ is a complementary pair of positively Borel sets.
- (iv) The statement: *for every closed set X , for all open sets Y, Z , if both $\{X, Y\}$ and $\{X, Z\}$ are complementary pairs, then $Y = Z$* , is not provable intuitionistically.

Proof: The proofs of (i) and (ii) are straightforward and left to the reader. (iii) is a consequence of earlier results, see Sections 4.6-10. It shows clearly that in the realm of positively Borel sets we do not have unicity of complements. (iv) claims that we

cannot prove unicity of complements for sets from the first level of the hierarchy. The statement mentioned is equivalent to the so-called *generalized Markov Principle*. This principle claims that, for every α , if $\neg\neg\exists n[\alpha(n) = 0]$, then $\exists n[\alpha(n) = 0]$. We do not see why this principle should be true, and do not want to use it. \square

7.3.2 Complementary pairs of positively Borel sets are considered by Brouwer, see [6], page 89, line 21-27, (he is studying $\mathbf{\Pi}_2^0$ and $\mathbf{\Sigma}_2^0$), and also, more generally by Martin-Löf in [14], page 80, and by Bishop and Bridges in [1], pages 73-75.

7.4 We define a function $\langle \rangle$ from $\mathcal{N} \times \mathcal{N}$ to \mathcal{N} such that for all α, β , $\langle \alpha, \beta \rangle^0 = \alpha$ and for each $n > 0$, $\langle \alpha, \beta \rangle^n = \beta$.

For every subset X of \mathcal{N} , for every α , we let $X \upharpoonright \alpha$ be the set of all β in \mathcal{N} such that $\langle \alpha, \beta \rangle$ belongs to X .

We introduce, for each non-zero stump σ , subsets US_σ and UP_σ of \mathcal{N} by means of the following definition.

- (i) US_{1^*} is the set of all α such that for some m, n , $\alpha^0(m) = \overline{\alpha^1}(n) + 1$.
 UP_{1^*} is the set of all α that do not belong to US_{1^*} .
- (ii) For every non-zero stump σ different from 1^* , US_σ is the set of all α such that for some n , $\langle \alpha^{0,n}, \alpha^1 \rangle$ belongs to UP_{σ^n} , and UP_σ is the set of all α such that for all n , $\langle \alpha^{0,n}, \alpha^1 \rangle$ belongs to US_{σ^n} .

We call US_σ and UP_σ the *universal* or *cataloguing* sets of level σ .

We also introduce, for each non-zero stump σ , subsets E_σ and A_σ of \mathcal{N} by means of the following inductive definition:

- (i) E_{1^*} is the set of all α such that for some n , $\alpha(\langle n \rangle) \neq 0$.
 A_{1^*} is the set of all α such that for all n , $\alpha(\langle n \rangle) = 0$.
- (ii) For every non-zero stump σ different from 1^* , E_σ is the set of all α such that for some n , α^n belongs to A_{σ^n} , and A_σ is the set of all α such that for all n , α^n belongs to E_{σ^n} .

We call E_σ and A_σ the *canonical complete* sets of level σ , and also the *leading sets* of $\mathbf{\Sigma}_\sigma^0$, $\mathbf{\Pi}_\sigma^0$, respectively.

7.4.1 Theorem:

- (i) For every non-zero stump σ , for all subsets X, Y of \mathcal{N} , if both X and Y belong to $\mathbf{\Sigma}_\sigma^0$, then $X \cap Y$ belongs to $\mathbf{\Sigma}_\sigma^0$, and: if both X and Y belong to $\mathbf{\Pi}_\sigma^0$, then $X \cap Y$ belongs to $\mathbf{\Pi}_\sigma^0$.
- (ii) For every non-zero stump σ , for every sequence X_0, X_1, \dots of subsets of \mathcal{N} , if, for each n , X_n belongs to $\mathbf{\Sigma}_\sigma^0$, then $\bigcup_{n \in \mathbb{N}} X_n$ belongs to $\mathbf{\Sigma}_\sigma^0$, and: if, for each n , X_n belongs to $\mathbf{\Pi}_\sigma^0$, then $\bigcap_{n \in \mathbb{N}} X_n$ belongs to $\mathbf{\Pi}_\sigma^0$.

- (iii) For every non-zero stump σ , for all subsets X, Y of \mathcal{N} , if Y belongs to Σ_σ^0 , and X reduces to Y , then X belongs to Σ_σ^0 , and: if Y belongs to Π_σ^0 and X reduces to Y , then X belongs to Π_σ^0 .
- (iv) For every non-zero hereditarily repetitive stump σ different from 1^* , for every subset X of \mathcal{N} :
 X belongs to Σ_σ^0 if and only if there exists a sequence X_0, X_1, \dots of subsets of \mathcal{N} such that for every n , X_n belongs to $\Pi_{\sigma^n}^0$, and $X = \bigcup_{n \in \mathbb{N}} X_n$,
and: X belongs to Π_σ^0 if and only if there exists a sequence X_0, X_1, \dots of subsets of \mathcal{N} such that for every n , X_n belongs to $\Sigma_{\sigma^n}^0$, and $X = \bigcap_{n \in \mathbb{N}} X_n$.
- (v) For every non-zero hereditarily repetitive stump σ , for every subset X of \mathcal{N} , X belongs to Σ_σ^0 if and only if, for some α , X coincides with $US_\sigma \upharpoonright \alpha$, and: X belongs to Π_σ^0 if and only if, for some α , X coincides with $UP_\sigma \upharpoonright \alpha$. In addition, $\{US_\sigma, UP_\sigma\}$ is a complementary pair of positively Borel sets.
- (vi) For every non-zero hereditarily repetitive stump σ , for every subset X of \mathcal{N} , X belongs to Σ_σ^0 if and only if X reduces to E_σ , and: X belongs to Π_σ^0 if and only if X reduces to A_σ .
In addition, $\{E_\sigma, A_\sigma\}$ is a complementary pair of positively Borel sets.

Proof: The proofs are straightforward and left to the reader. One sometimes has to use the Second Axiom of Countable Choice. \square

7.4.2 As a further remark we may add that for all non-zero stumps σ, τ , Σ_σ^0 forms part of Σ_τ^0 if and only if Π_σ^0 forms part of Π_τ^0 if and only if $\sigma \leq \tau$. Also, for every non-zero stump σ we may find a hereditarily repetitive non-zero stump τ such that both $\sigma \leq \tau$ and $\tau \leq \sigma$, and therefore $\Sigma_\sigma^0 = \Sigma_\tau^0$ and $\Pi_\sigma^0 = \Pi_\tau^0$. So it does no harm, in this context, to restrict attention to hereditarily repetitive stumps. Observe that, for each positive n , the stump n^* is hereditarily repetitive.

7.5 For every α , for every a , we define an infinite sequence ${}^a\alpha$, as follows, by induction on $\text{length}(a)$: ${}^{\langle \rangle}\alpha := \alpha$ and for all a, n : ${}^{a*\langle n \rangle}\alpha := ({}^a\alpha)^n$. So for every α, a, m one has ${}^a\alpha(m) = \alpha(a * m)$.

For all a, b we define: A does not compare with b , or a, b are incompatible, notation $a \perp b$, if and only if there is no m such that either $a * m = b$ or $b * m = a$.

Observe that for every non-zero stump σ , for every a , one may decide if there exists n such that $a * \langle n \rangle$ belongs to σ or not. If there is no such n , we say that a is a *final position* in σ .

For every stump σ , for every a , we say that a is *just outside* σ if and only if there exists b, n such that $a = b * \langle n \rangle$ and b belongs to σ while a does not. For every stump σ , for every a , one may decide if a is just outside σ or not.

7.5.1 Theorem: (*The Classical Borel Hierarchy Theorem*)

- (i) For every non-zero hereditarily repetitive stump σ , if either Π_σ^0 forms part of Σ_σ^0 , or Σ_σ^0 forms part of Π_σ^0 , then there exists γ belonging to neither one of US_σ, UP_σ .
- (ii) For every function f from \mathcal{N} to \mathcal{N} there exists α in \mathcal{N} such that α belongs to E_1 if and only if $f|\alpha$ belongs to E_1 . For every decidable subset A of \mathbb{N} consisting of mutually incompatible numbers, for every function f from \mathcal{N} to \mathcal{N} there exists α such that for each a in A , ${}^a\alpha$ belongs to E_1 if and only if ${}^a(f|\alpha)$ belongs to E_1 .
- (iii) For every non-zero stump σ , for every function f from \mathcal{N} to \mathcal{N} , there exists α such that α belongs to E_σ if and only if $f|\alpha$ belongs to E_σ , and: α belongs to A_σ if and only if $f|\alpha$ belongs to A_σ .
- (iv) For every non-zero hereditarily repetitive stump σ , if either A_σ reduces to E_σ or E_σ reduces to A_σ , then there exists α belonging to neither one of A_σ, E_σ .

Proof: (i) For every non-zero hereditarily repetitive stump σ , we let DS_σ be the set of all α such that $\langle \alpha, \alpha \rangle$ belongs to US_σ , and we let DP_σ be the set of all α such that $\langle \alpha, \alpha \rangle$ belongs to UP_σ . We call DS_σ and DP_σ the *diagonal sets of level σ* .

Observe that DS_σ, DP_σ belong to $\Sigma_\sigma^0, \Pi_\sigma^0$, respectively. Suppose, for instance, that Π_σ^0 forms part of Σ_σ^0 , then in particular DP_σ belongs to Σ_σ^0 , and we may find β such that DP_σ coincides with $US_\sigma \upharpoonright \beta$, and therefore for every α , $\langle \alpha, \alpha \rangle$ belongs to UP_σ if and only if $\langle \beta, \alpha \rangle$ belongs to US_σ . Define $\gamma := \langle \beta, \beta \rangle$ and observe that γ cannot belong to either UP_σ or US_σ .

(ii) Let f be a function from \mathcal{N} to \mathcal{N} . We define an infinite sequence α , by induction, as follows. Let n be a natural number and suppose we decided already on $\alpha(0), \dots, \alpha(n-1)$. We now consider if there exist $i, j < n$ such that $f^i(\overline{\alpha}j) > 1$, whereas, for each every $q < j$, $f^i(\overline{\alpha}q) = 0$. If so, we define $\alpha(n) := 1$, if not, we define $\alpha(n) := 0$. It is not difficult to see that α belongs to E_1 if and only if $f|\alpha$ belongs to E_1 . Now let A be a decidable subset of \mathbb{N} consisting of mutually incompatible natural numbers and let f be a function from \mathcal{N} to \mathcal{N} . We define an infinite sequence α , by induction, as follows. Let n be a natural number and suppose we decided already on $\alpha(0), \dots, \alpha(n-1)$. We now consider if there exist a in A , k in \mathbb{N} such that $n = a * k$ and for some $i, j < n$, $f^{a*i}(\overline{\alpha}j) > 1$, whereas, for every $q < j$, $f^{a*i}(\overline{\alpha}q) = 0$. If so, we define $\alpha(n) := 1$, if not, we define $\alpha(n) := 0$. Observe that for each a in A , there exists k such that $\alpha(a * k) = 1$ if and only if there exists i such that $(f|\alpha)(a * i) \neq 0$, that is ${}^a\alpha$ belongs to E_1 if and only if ${}^a(f|\alpha)$ belongs to E_1 .

(iii) Let σ be a non-zero stump and let A be the set of all final positions in σ . Let F be a function from \mathcal{N} to \mathcal{N} . Using (ii), construct α such that for every a in A , ${}^a\alpha$ belongs to E_1 if and only if ${}^a(f|\alpha)$ belongs to E_1 . Using the Principle of Stump Induction, see Section 1.5.4, prove that for every a belonging to σ , both ${}^a\alpha$ belongs to $E_{(a\sigma)}$ if and only if ${}^a(f|\alpha)$ belongs to $E_{(a\sigma)}$ and: ${}^a\alpha$ belongs to $A_{(a\sigma)}$ if and only if ${}^a(f|\alpha)$ belongs to $A_{(a\sigma)}$. In particular: α belongs to E_σ if and only if $f|\alpha$ belongs to

E_σ and: α belongs to A_σ if and only if $f|\alpha$ belongs to A_σ .

(iv) Let σ be a non-zero hereditarily repetitive stump and let f be a function from \mathcal{N} to \mathcal{N} reducing A_σ to E_σ , that is, for every α , α belongs to A_σ if and only if $f|\alpha$ belongs to E_σ . Using (iii), construct α such that α belongs to A_σ if and only if $f|\alpha$ belongs to A_σ and: α belongs to E_σ if and only if $f|\alpha$ belongs to E_σ . If α should belong to A_σ , $f|\alpha$ would belong to both E_σ and A_σ , contradiction. If α should belong to E_σ , $f|\alpha$ would belong to E_σ and therefore α would belong both to A_σ and E_σ , contradiction. Therefore α belongs to neither one of A_σ , E_σ . \square

7.5.2 The second statement of Theorem 7.5.1 is a kind of fixed point theorem and the argument given may be compared to the argument developed by S.C. Kleene for his First Recursion Theorem.

The reader will perhaps be surprised by the careful formulation of the first and fourth statement of Theorem 7.5.1. Does not the assumption that, for some non-zero stump σ , some α does not belong to either one of A_σ , E_σ , lead to a contradiction?

In fact, we are only entitled to draw this conclusion in case $\sigma = 1^*$. It thus follows from Theorem 7.5.1 that Π_1^0 is not included in Σ_1^0 and that Σ_1^0 is not included in Π_1^0 but that is all.

One might hope for the conclusion that the classes Π_2^0 and Σ_2^0 are not included in each other, but this hope realizes only if one makes some unfounded assumption like the *generalized Markov Principle*: for every α , if $\neg\neg\exists n[\alpha(n) = 0]$, then $\exists n[\alpha(n) = 0]$, or *S. Kuroda's axiom*: for every subset P of \mathbb{N} , if $\forall n[\neg\neg P(n)]$, then $\neg\neg\forall n \in \mathbb{N}[P(n)]$. It is not true, although stated in [17] and [19], that assuming Markov's Principle enables one to climb all further steps of the hierarchy: already the third level is still out of reach.

7.5.3 We add an example showing that in intuitionistic mathematics it is possible that statements $\neg\exists x\forall y\exists z[P(x, y, z)]$ and $\neg\forall x\exists y\forall z[\neg P(x, y, z)]$ and $\forall x\forall y\forall z[P(x, y, z) \vee \neg P(x, y, z)]$ are simultaneously true. We claim that (i) $\neg\exists\alpha\forall n\exists m[\alpha(n) = 0 \wedge \alpha(m) \neq 0]$ and (ii) $\neg\forall\alpha\exists n\forall m[\alpha(n) \neq 0 \vee \alpha(m) = 0]$ and (iii) $\forall\alpha\forall n\forall m[(\alpha(n) = 0 \wedge \alpha(m) \neq 0) \vee (\alpha(n) \neq 0 \vee \alpha(m) = 0)]$.

We only prove (ii). Assume $\forall\alpha\exists n\forall m[\alpha(n) \neq 0 \vee \alpha(m) = 0]$.

Using the Continuity Principle, we find n, p such that for every α passing through $\bar{0}p$ either $\alpha(n) \neq 0$ or $\alpha = \bar{0}$. Now consider $\alpha = \bar{0}q * \underline{1}$ where q is greater than both n, p . Contradiction.

This example shows that it is impossible to obtain from Theorem 7.5.1 the conclusion that Π_3^0 is not included in Σ_3^0 , if one only uses the rules of intuitionistic logic and no further mathematical assumptions.

7.6 We now want to prove the finite case of the Intuitionistic Borel Hierarchy Theorem.

Given any subset X of \mathcal{N} we let the *infinite disjunction* of X , notation $D^\omega(X)$, be the set of all α such that, for some n , α^n belongs to X , and we let the *infinite conjunction* of X , notation $C^\omega(X)$, be the set of all α such that, for all n , α^n belongs to X .

A subset Y of \mathcal{N} reduces to $D^\omega(X)$, $C^\omega(X)$ respectively, if and only if there exists a sequence Y_0, Y_1, \dots of subsets of \mathcal{N} , each of them reducing to X with the property that the set Y coincides with the set $\bigcup_{n \in \mathbb{N}} Y_n, \bigcap_{n \in \mathbb{N}} Y_n$, respectively.

Observe that, for each positive n , the set $D^\omega(A_n)$ coincides with the set E_{n+1} and the set $C^\omega(E_n)$ coincides with the set A_{n+1} .

Let (X, Y) be a pair of sets such that every element of X is apart from every element of Y . We say that X is *strongly irreducible* to Y if and only if for every function f from \mathcal{N} to \mathcal{N} mapping X into Y there exists also α in Y such that $f|\alpha$ belongs to Y . We say that X is *very strongly irreducible* to Y if and only if for every decidable subset A of \mathbb{N} consisting of mutually incompatible numbers, (that is, for all a, b in A , if $a \neq b$, then $a \perp b$), for every function f from \mathcal{N} to \mathcal{N} such that, for every a in A , for every α , if ${}^a\alpha$ belongs to X , then ${}^a(f|\alpha)$ belongs to Y , there exists α such that for every a in A both ${}^a\alpha$ and ${}^a(f|\alpha)$ belong to Y . Observe that, if X is very strongly irreducible to Y , then it is also true that for every decidable subset A of \mathbb{N} consisting of mutually incompatible numbers, for every ρ in \mathcal{N} , for every function f from \mathcal{N} to \mathcal{N} with the property that for all a in A , for all α , if ${}^a\alpha$ belongs to X , then $\rho^{(a)}(f|\alpha)$ belongs to Y , there exists α such that for all a in A , both ${}^a\alpha$ and $\rho^{(a)}(f|\alpha)$ belong to Y .

Let X be a subset of \mathcal{N} . X is called *strictly analytic* if and only if there exists a function γ from \mathcal{N} to \mathcal{N} such that X coincides with the range of γ , so for every α , α belongs to X if and only if, for some β , β coincides with $\gamma|\alpha$.

7.6.1 Theorem:

The set A_1 is very strongly irreducible to the set E_1 and the set E_1 is very strongly irreducible to the set A_1 .

Proof: Let A be a decidable subset of \mathbb{N} consisting of mutually incompatible numbers and let f be a function from \mathcal{N} to \mathcal{N} such that for all a in A , for all α , if ${}^a\alpha$ belongs to A_1 , then ${}^a(f|\alpha)$ belongs to E_1 . Using Theorem 7.5.1(ii), find α such that for each a in A , ${}^a\alpha$ belongs to E_1 if and only if ${}^a(f|\alpha)$ belongs to E_1 . We claim that for each a in A both ${}^a\alpha$ and ${}^a(f|\alpha)$ belong to E_1 . Let a belong to A and let β be such that ${}^a\beta = \underline{0}$ and for each i , if there is no j such that $i = a * j$, then $\beta(i) = \alpha(i)$. Calculate q such that $({}^a(f|\beta))(q) \neq 0$, and find p such that $f^{a*q}(\bar{\beta}p) > 1$ and, for every $j < p$, $f^{a*(q)}(\bar{\beta}j) = 0$. Now distinguish two cases. Either $\bar{\alpha}p = \bar{\beta}p$, therefore ${}^a(f|\alpha)$ and also ${}^a\alpha$ belong to E_1 , or $\bar{\alpha}p \neq \bar{\beta}p$ and therefore ${}^a\alpha$ and also ${}^a(f|\alpha)$ belong to E_1 .

The proof of the second statement is similar. \square

7.6.2 Lemma: (First Continuity Lemma)

Let X be a strictly analytic subset of \mathcal{N} .

Let R be a subset of $\mathcal{N} \times \mathbb{N}$ and a a natural number such that for every α , if ${}^a\alpha$ belongs to $C^\omega(X)$, then there exists m such that αRm . Then:

- (i) For every α such that ${}^a\alpha$ belongs to $C^\omega(X)$, there exists n, m with the property that for every β such that ${}^a\beta$ belongs to $C^\omega(X)$, if $\bar{\alpha}n = \bar{\beta}n$ and for all $j < n$, ${}^{a*(j)}\alpha = {}^{a*(j)}\beta$, then βRm .
- (ii) For every γ in $C^\omega(X)$ there exist functions μ, ν from \mathcal{N} to \mathbb{N} such that for every α such that ${}^a\alpha$ equals γ , for every β such that ${}^a\beta$ belongs to $C^\omega(X)$, if both $\bar{\alpha}\nu(\alpha) = \bar{\beta}\nu(\alpha)$ and for all $j < \nu(\alpha)$, ${}^{a*(j)}\beta = \gamma^j$, then $\beta R\mu(\alpha)$.

Proof: Let f be a function from \mathcal{N} to \mathcal{N} such that X coincides with the range of f . Now let g be a function from \mathcal{N} to \mathcal{N} such that for every α , $(g|\alpha)(0) = \alpha(0)$ and for each n , $(g|\alpha)^n = f|(\alpha^n)$. Observe that $C^\omega(X)$ coincides with the range of g . Let h be a function from \mathcal{N} to \mathcal{N} such that for every α , ${}^a(h|\alpha) = g|(\alpha^1)$ and for each n , if there is no j such that $n = a * j$, then $(h|\alpha)(n) = \alpha^0(n)$. Observe that the set of all α such that ${}^a\alpha$ belongs to $C^\omega(X)$ coincides with the range of h .

Assume that for every α such that ${}^a\alpha$ belongs to $C^\omega(X)$ there exists m such that αRm . Then for every β there exists m such that $(h|\beta)Rm$.

(i) Assume that we have some α such that ${}^a\alpha$ belongs to $C^\omega(X)$.

Find γ such that $\alpha = h|\gamma$, and using the Continuity Principle, find m, n such that for every δ , if $\bar{\gamma}n = \bar{\delta}n$, then $(h|\delta)Rm$.

Observe that for every β such that ${}^a\beta$ belongs to $C^\omega(X)$ and $\bar{\alpha}n = \bar{\beta}n$ and for all $j < n$, $\alpha^{a*(j)}\beta$ there exists δ such that $\beta = h|\delta$ and $\bar{\gamma}n = \bar{\delta}n$.

(ii) Using the First Axiom of Continuous Choice, determine functions π, ρ from \mathcal{N} to \mathbb{N} such that for every β , $(h|\beta)R\pi(\beta)$ and $\rho(\beta) := \mu n[\pi(\bar{\beta}n) \neq 0] + 1$. Observe that for every β , for every α , if ${}^a\alpha$ belongs to $C^\omega(X)$ and both $\bar{\alpha}\rho(\beta) = \bar{\beta}\rho(\beta)$ and for each $j < \rho(\beta)$, ${}^{a*(j)}\alpha = {}^{a*(j)}(h|\beta)$, there will exist δ passing through $\bar{\beta}\rho(\beta)$ such that $\alpha = h|\beta$, and therefore $\alpha R\pi(\beta)$. Let γ belong to $C^\omega(X)$. Construct a function η from \mathcal{N} to \mathcal{N} such that for each α such that ${}^a\alpha = \gamma$, the sequence $h|(\eta|\alpha)$ coincides with α . Define functions μ, ν from \mathcal{N} to \mathbb{N} by: for all α , $\mu(x) := \pi|(\eta|\alpha)$ and $\nu(\alpha) := \rho|(\eta|\alpha)$. One easily verifies that μ, ν satisfy the requirements. \square

7.6.3 Theorem: Let X, Y be strictly analytic subsets of \mathcal{N} such that every element of X is apart from every element of Y .

- (i) If X is very strongly irreducible to Y , then $D^\omega(X)$ is very strongly irreducible to $C^\omega(Y)$.
- (ii) If X is very strongly irreducible to Y , then $C^\omega(X)$ is very strongly irreducible to $D^\omega(Y)$.

Proof: (i) Let A be a decidable subset of \mathbb{N} consisting of mutually incompatible numbers and let f be a function from \mathcal{N} to \mathcal{N} such that for every a in A , for every

α , if ${}^a\alpha$ belongs to $D^\omega(X)$, then ${}^a(f|\alpha)$ belongs to $C^\omega(Y)$. It follows that for every a in A , for every n , for every α , if ${}^{a* \langle n \rangle}\alpha$ belongs to X , then ${}^{a* \langle n \rangle}(f|\alpha)$ belongs to Y . Observe that the set of all numbers $a * \langle n \rangle$, where a belongs to A and n to \mathbb{N} is a decidable set of mutually incompatible numbers. We use the fact that X is very strongly irreducible to Y and find some α such that for all a in A , for all n , both ${}^{a* \langle n \rangle}\alpha$ and ${}^{a* \langle n \rangle}(f|\alpha)$ belong to Y , and therefore, for all a in A , both ${}^a\alpha$ and ${}^a(f|\alpha)$ belong to $C^\omega(Y)$.

(ii) Let A be a decidable subset of \mathbb{N} consisting of mutually incompatible numbers and let f be a function from \mathcal{N} to \mathcal{N} such that for every a in A , for every α , if ${}^a\alpha$ belongs to $C^\omega(X)$, then ${}^a(f|\alpha)$ belongs to $D^\omega(Y)$. Let γ be an element of $C^\omega(X)$. Using Lemma 7.6.2(ii) we find for each a in A functions μ_a and ν_a from \mathcal{N} to \mathbb{N} such that for every α such that ${}^a\alpha$ equals γ , for every β such that ${}^a\beta$ belongs to $C^\omega(X)$, if both $\overline{\alpha}\nu_a(\alpha) = \overline{\beta}\nu_a(\alpha)$ and for all $j < \nu_a(\alpha)$, ${}^{a* \langle j \rangle}\beta$ equals γ^j , then ${}^{a* \langle \mu_a(\alpha) \rangle}(f|\beta)$ belongs to Y .

We now define a sequence g_0, g_1, \dots of functions from \mathcal{N} to \mathcal{N} such that, for each β , the sequence $g_0|\beta, g_1|\beta, \dots$ is a convergent sequence of elements of \mathcal{N} .

We let α_0 be an element of \mathcal{N} with the property that for each a in A , ${}^a(\alpha_0)$ equals γ , and define, for each β , $g_0|\beta := \alpha_0$. Now assume that a belongs to \mathbb{N} and that g_a has been defined. If a does not belong to A , we define: $g_{a+1} := g_a$.

If a belongs to A we let p_a be the largest of all numbers $\nu_b(g_b|\beta)$, b in A , $b \leq a$. We then define the function g_{a+1} from \mathcal{N} to \mathcal{N} such that for every β , the sequence ${}^{a* \langle p_a \rangle}g_{a+1}(\beta)$ equals β and for all j , if there is no i such that $j = a * \langle p_a \rangle * i$, then $(g_{a+1}(\beta))(j) = (g_a(\beta))(j)$.

It will be clear that, for each β , the sequence $g_0|\beta, g_1|\beta, \dots$ converges. We define the function g from \mathcal{N} to \mathcal{N} by: for each β , $g|\beta := \lim_{a \rightarrow \infty} g_a|\beta$.

Observe that for each β , for each a in A , if ${}^a\beta$ belongs to X , then ${}^{a* \langle p_a \rangle}(g|\beta)$ belongs to X , and ${}^a(g|\beta)$ belongs to $C^\omega(X)$, and ${}^{a* \langle \mu_a(g_a|\beta) \rangle}(f|(g|\beta))$ belongs to Y . So we may determine β such that for all a in A , both ${}^{a* \langle p_a \rangle}(g|\beta)$ and ${}^{a* \langle \mu_a(g_a|\beta) \rangle}(f|(g|\beta))$ belong to Y , therefore both ${}^a(g|\beta)$ and ${}^a(f|(g|\beta))$ belong to $D^\omega(Y)$. \square

7.6.4 Theorem: (Finite Borel Hierarchy Theorem)

- (i) For each n , for every function f from \mathcal{N} to \mathcal{N} mapping A_n into E_n , there exists α such that both α and $f|\alpha$ belong to E_n .
- (ii) For each n , for every function f from \mathcal{N} to \mathcal{N} mapping E_n into A_n , there exists α such that both α and $f|\alpha$ belong to A_n .

Proof: Observe that, for each n , (A_n, E_n) is a complementary pair of positively Borel and strictly analytic sets. Using Theorems 7.6.1 and 7.6.3 conclude that, for each n , the set A_n is very strongly irreducible to the set E_n and the set E_n is very strongly irreducible to the set A_n . \square

7.7 We intend to prove the Intuitionistic Borel Hierarchy Theorem itself.

7.7.1 For every non-zero stump σ , for every α , we introduce a game $G(\sigma, \alpha)$ for players I, II. It is a game of perfect information. Player I starts and chooses a natural number n_0 , then player II chooses a natural number n_1 , and so they continue choosing alternately a natural number. The play ends as soon as the number $\langle n_0, \dots, n_{k-1} \rangle$ is just outside σ . Player I is the *winner* if and only if either k is odd and $\alpha(\langle n_0, \dots, n_{k-1} \rangle)$ differs from 0, or k is even and $\alpha(\langle n_0, \dots, n_{k-1} \rangle)$ equals 0. We then say that the number $\langle n_0, \dots, n_{k-1} \rangle$ is a *win for player I in the game $G(\sigma, \alpha)$* . Player II is the *winner* if and only if player I is not. In that case the number $\langle n_0, \dots, n_{k-1} \rangle$ is called a *win for player II in the game $G(\sigma, \alpha)$* . An element γ of \mathcal{N} may be thought of as a strategy for either player I or player II, as follows. For every $s = \langle (s)_0, \dots, (s)_{k-1} \rangle$ where $k = \text{length}(s)$, for every γ , we define: s I-obey γ , or γ I-govern s , if and only if, for every i , if $2i < k$, then $(s)_{2i} = \gamma(\langle (s)_1, (s)_3, \dots, (s)_{2i-1} \rangle)$ and: s II-obey γ , or γ II-govern s , if and only if, for every i , if $2i + 1 < k$, then $(s)_{2i+1} = \gamma(\langle (s)_0, (s)_2, \dots, (s)_{2i} \rangle)$. Suppose that for some non-zero stump σ , for some γ, α , every position just outside σ I-obeying to γ is a win for player I in the game $G(\sigma, \alpha)$. We then say that γ is a *winning strategy for player I in the game $G(\sigma, \alpha)$* . Also, if every position just outside σ II-obeying to γ is a win for player II in the game $G(\sigma, \alpha)$, we say that γ is a *winning strategy for player II in the game $G(\sigma, \alpha)$* .

For every γ, α , for every non-zero stump σ , we define two elements of \mathcal{N} , $\text{Corr}_I^\sigma(\gamma, \alpha)$ and $\text{Corr}_{II}^\sigma(\gamma, \alpha)$, as follows. For every s , if s is not a position just outside σ I-obeying γ , then $(\text{Corr}_I^\sigma(\gamma, \alpha))(s)$ equals $\alpha(s)$, but if s is a position just outside σ I-obeying γ , then, if $\text{length}(s)$ is odd, $(\text{Corr}_I^\sigma(\gamma, \alpha))(s)$ equals the larger one of the numbers 1, $\alpha(s)$, and if $\text{length}(s)$ is even, $(\text{Corr}_I^\sigma(\gamma, \alpha))(s)$ equals 0. Also, for every s , if s is not a position just outside σ II-obeying γ , then $(\text{Corr}_{II}^\sigma(\gamma, \alpha))(s)$ equals $\alpha(s)$, but if s is a position just outside σ II-obeying γ , then, if $\text{length}(s)$ is odd, $(\text{Corr}_{II}^\sigma(\gamma, \alpha))(s)$ equals 0, and if $\text{length}(s)$ is even, $(\text{Corr}_{II}^\sigma(\gamma, \alpha))(s)$ equals the larger one of the numbers 1, $\alpha(s)$.

We might pronounce “ $\text{Corr}_I^\sigma(\gamma, \alpha)$ ” as: “ α -as-corrected-according-to- γ -as a strategy for player I in the game $G(\sigma, \alpha)$.”

7.7.2 Theorem:

- (i) For every non-zero stump σ , for every α , $E_\sigma(\alpha)$ if and only if there exists γ such that γ is a winning strategy for player I in the game $G(\sigma, \alpha)$ and α equals $\text{Corr}_I^\sigma(\gamma, \alpha)$.
- (ii) For every non-zero stump σ , for every α , $A_\sigma(\alpha)$ if and only if there exists γ such that γ is a winning strategy for player II in the game $G(\sigma, \alpha)$ and α equals $\text{Corr}_{II}^\sigma(\gamma, \alpha)$.

Proof: The proof is straightforward and uses the Second Axiom of Countable Choice.

⊠

7.7.3 We want to show that for every non-zero hereditarily repetitive stump σ , for every function f from \mathcal{N} to \mathcal{N} mapping A_σ into E_σ , there exists α such that both α itself and its image $f|\alpha$ belong to E_σ , that is, there exist γ, δ such that $\alpha = \text{Corr}_I^\sigma(\gamma, \alpha)$ and $f|\alpha = \text{Corr}_I^\sigma(\delta, f|\alpha)$. With this aim in mind we introduce some further notions.

Let C be a spread. We say that C is a *value-dictating spread* if there exist a decidable subset A of \mathbb{N} and an element α of \mathcal{N} such that for every β , β belongs to C if and only if for each n in A , $\beta(n) = \alpha(n)$.

For every spread C , for all m, n , we say that C *dictates the value m in n* if and only if, for all α in C , $\alpha(m) = n$. We say that m is *without choice in C* if, for some n , C dictates the value n in C . We say that m is *free in C* if and only if for all α in C , for all β , if for every $j \neq m$, $\alpha(j) = \beta(j)$, then also β belongs to C . We say that m is *completely free in C* if and only if, for all p , the number $m * p$ is free in C . We say that m is *almost completely free in C* if and only if for all but finitely many numbers p , the number $m * p$ is free in C , and for all p , either $m * p$ is free in C , or C dictates the value 0 in $m * p$.

Let C be a spread. We define the *minimal element* of C , notation $\text{Min}(C)$, as follows: for each n , $(\text{Min}(C))(n) :=$ the least k such that some element of C passes through $\overline{\text{Min}(C)}n * \langle k \rangle$.

7.7.4 Lemma: (*First Basic Lemma*)

Let C be a value-dictating spread and s a natural number such that s is almost completely free in C .

Let f be a function from \mathcal{N} to \mathcal{N}

Suppose that for all α in C , if ${}^s\alpha$ belongs to A_1 , then ${}^s(f|\alpha)$ belongs to E_1 .

There exists a value-dictating subsread D of C such that for all α in D , both ${}^s\alpha$ and ${}^s(f|\alpha)$ belong to E_1 , and for all t such that $t \perp s$, if t is almost completely free in C then t is almost completely free in D .

Proof: Let $\text{Min}(C)$ be the minimal element of C . Observe that ${}^s\text{Min}(C)$ belongs to A_1 and find n such that ${}^s(f|\text{Min}(C))(\langle n \rangle) \neq 0$. Calculate m such that for every α in C , if $\overline{\alpha}m = \overline{\text{Min}(C)}m$, then ${}^s(f|\alpha)(\langle n \rangle) = {}^s(f|\text{Min}(C))(\langle n \rangle)$. Let D be the set of all α in C such that $\overline{\alpha}m = \overline{\text{Min}(C)}m$ and ${}^s\alpha = \overline{0}m * \underline{1}$.

It is easy to see that D satisfies the requirements. ⊠

7.7.5 Lemma: (*Second Basic Lemma*)

Let C be a value-dictating spread and s a natural number such that s is almost completely free in C .

Let f be a function from \mathcal{N} to \mathcal{N} .

Suppose that for all α in C , if ${}^s\alpha$ belongs to E_1 , then ${}^s(f|\alpha)$ belongs to A_1 .

There exists a value-dictating subspread D of C such that for all α in D , both ${}^s\alpha$ and ${}^s(f|\alpha)$ belong to A_1 , and for all t such that $t \perp s$, if t is almost completely free in C , then t is almost completely free in D .

Proof: Let D be the set of all α in C such that ${}^s\alpha = \underline{0}$. It is easy to see that D satisfies the requirements. \square

7.7.6 Lemma: (*Second Continuity Lemma*)

Let C be a value-dictating spread and s a natural number such that s is almost completely free in C .

Let τ be a non-zero stump different from 1^* and let R be a subset of $\mathcal{N} \times \mathbb{N}$, such that for every α in C , if ${}^s\alpha$ belongs to A_τ , then there exists m such that $\alpha R m$.

Then, given any α in C such that ${}^s\alpha$ belongs to A_τ we may calculate m, n such that for every β in C such that ${}^s\beta$ belongs to A_τ , if $\bar{\alpha}n = \bar{\beta}n$ and for every $j < n$, ${}^{s*(j)}\alpha = {}^{s*(j)}\beta$, then $\beta R m$.

Proof: Assume α belongs to C and ${}^s\alpha$ belongs to A_τ . We calculate γ such that ${}^s\alpha$ coincides with $\text{Corr}_{II}^\tau(\gamma, {}^s\alpha)$, and p such that for every $q > p$, $s * q$ is free in C . We let X be the set of all δ such that for every $q \leq p$, $\delta^q = \gamma^q$. We define a function h from $X \times C$ to C , as follows. For every δ in X , every β in C we require that ${}^s(h(\delta, \beta))$ equals $\text{Corr}_{II}^\tau(\delta, {}^s\beta)$, and for every t , if there is no j such that $t = s * j$, then $(h(\delta, \beta))(t) = \beta(t)$.

Observe that both X and C are spreads.

Applying the Continuity Principle we find m, n such that $n > p$ and for all δ in X , for all β in C , if $\bar{\beta}n = \bar{\alpha}n$ and $\bar{\delta}n = \bar{\gamma}n$, then $(h(\delta, \beta)) R m$.

Now assume that β belongs to C , and ${}^s\beta$ belongs to A_τ , and $\bar{\beta}n = \bar{\alpha}n$, and for all $j < n$, ${}^{s*(j)}\beta$ coincides with ${}^{s*(j)}\alpha$. Calculate δ in X such that $\beta = h(\delta, \beta)$ and $\bar{\delta}n = \bar{\gamma}n$, and conclude $\beta R m$. \square

7.7.7 Lemma: (*Main Lemma*)

Let C be a value-dictating spread and s a natural number such that s is almost completely free in C .

Let τ be a non-zero hereditarily repetitive stump different from 1^* , and let g be a function from \mathcal{N} to \mathcal{N} such that for all α in C , if ${}^s\alpha$ belongs to A_τ , then $g|\alpha$ belongs to E_τ .

There exist m, p and a value-dictating subspread D of C such that $\tau^m = \tau^p$, and $s * \langle m \rangle$ is completely free in D , and for all α in D , if ${}^{s*(m)}\alpha$ belongs

to E_{τ^m} , then ${}^s\alpha$ belongs to A^τ and $(g|\alpha)^p$ belong to $A_{\tau^p} = A_{\tau^m}$, and for all t such that $t \perp s$, if t is almost completely free in C , then t is almost completely free in D .

Proof: Let α be some element of C such that ${}^s\alpha$ belongs to A_τ .

Applying the previous Lemma, we find p, n such that for every β in C , if ${}^s\beta$ belongs to A_σ and $\bar{\beta}n = \bar{\alpha}n$ and for all $j < n$, ${}^{s*(j)}\beta$ coincides with ${}^{s*(j)}\alpha$, then $(g|\beta)^p$ belongs to A_{τ^p} . We now determine m such that $m > n$ and $\tau^p = \tau^m$ and $s*(m)$ is completely free in C and let D be the set of all β in C such that $\bar{\beta}m = \bar{\alpha}m$ and for all j , if $j \neq m$, then ${}^{s*(j)}\beta = {}^{s*(j)}\alpha$. D is easily seen to be a spread satisfying all our requirements. \square

7.7.8 Let s, c be natural numbers. s I-obey c if and only if for each i , if $2i < \text{length}(s)$, then $m := \langle (s)_1, (s)_3, \dots, (s)_{2i-1} \rangle$ is smaller than $\text{length}(c)$ and $(s)_{2i}$ equals $(c)_m$. Let s, t be natural numbers of equal length. We say that s is II-similar to t if and only if for each odd $i < \text{length}(s)$, $(s)_i = (t)_i$. If s is II-similar to t , then player II made the same moves in the course of reaching the position s as he made in the course of reaching the position t .

7.7.9 Theorem: (*Intuitionistic Borel Hierarchy Theorem, First Part*)

Let σ be a non-zero hereditarily repetitive stump and let f be a function from \mathcal{N} to \mathcal{N} mapping A_σ into E_σ .

There exists α in \mathcal{N} such that both α itself and $f|\alpha$ belong to E_σ .

Proof: We intend to build α, γ, δ such that α coincides with $\text{Corr}_I^\sigma(\gamma, \alpha)$ and $f|\alpha$ coincides with $\text{Corr}_I^\sigma(\delta, f|\alpha)$.

We define the sequences γ, δ step by step, first $\gamma(0), \delta(0)$, then $\gamma(1), \delta(1), \dots$, and at the same time we define a sequence C_0, C_1, \dots of value-dictating spreads such that the following conditions are satisfied:

- (1) for each n , for each pair $\langle s, t \rangle$ of II-similar positions of equal length, if s belongs to $\mathbb{B}(\sigma)$ and s I-obey $\bar{\gamma}n$ and t I-obey $\bar{\delta}n$, then also t belongs to $\mathbb{B}(\sigma)$ and ${}^s\sigma = {}^t\sigma$ and one of the four following cases obtains:
 - (i) s is a final position in σ of even length, and for all α in C_n , both ${}^s\alpha$ and ${}^t(f|\alpha)$ belong to A_1 .
 - (ii) s is a final position in σ of odd length, and for all α in C_n , both ${}^s\alpha$ and ${}^t(f|\alpha)$ belong to E_1 .
 - (iii) s is a non-final position in σ of even length, and for all α in C_n , if ${}^s\alpha$ belongs to $A_{(s\sigma)}$, then ${}^t(f|\alpha)$ belongs to $A_{(t\sigma)} = A_{(s\sigma)}$.
 - (iv) s is a non-final position in σ of odd length, and for all α in C_n , if ${}^s\alpha$ belongs to $E_{(s\sigma)}$, then ${}^t(f|\alpha)$ belongs to $E_{(t\sigma)} = E_{(s\sigma)}$.

- (2) For each n , each non-final position s of σ obeying $\bar{\gamma}n$ is almost free in C_n .
- (3) For each n , C_{n+1} is a subsread of C_n and each $i < n$ is without choice in C_n .

We define $C_0 = \mathcal{N}$. Observe that the empty position $0 = \langle \rangle$ is the only position I-obeying $0 = \bar{\gamma}0 = \bar{\delta}0$. Remark that for all α in C_0 , if ${}^0\alpha$ belongs to A_σ , then ${}^0(f|\alpha)$ belongs to E_σ .

Now assume that, for some n , $C_n, \bar{\gamma}n, \bar{\delta}n$ have been defined and the conditions (1)-(3) are satisfied so far.

We determine $k = \text{length}(n)$ and consider the finite sequence coded by n , $n = \langle (n)_0, \dots, (n)_{k-1} \rangle$. The elements of this sequence have to be thought of as moves by player II. We consider

$$s := \langle \gamma(0), (n)_0, \gamma(\langle (n)_0 \rangle), \dots, \gamma(\langle (n)_0, \dots, (n)_{k-2} \rangle), (n)_{k-1} \rangle.$$

Observe that $\text{length}(s) = 2k$. We also consider

$$t := \langle \delta(0), (n)_0, \delta(\langle (n)_0 \rangle), \dots, \delta(\langle (n)_0, \dots, (n)_{k-2} \rangle), (n)_{k-1} \rangle.$$

Observe that s is II-similar to t , and s obeys $\bar{\gamma}n$, and t obeys $\bar{\delta}n$.

We now distinguish several cases.

- (*) s does not belong to σ .

We now define: $\gamma(n) := 0$ and $\delta(n) := 0$ and C_{n+1} is the set of all α in C_n passing through $\overline{\text{Min}(C_n)}(n+1)$.

- (**) s is a final position in σ .

We then know: for all α in C_n , if ${}^s\alpha$ belongs to A_1 , then ${}^t(f|\alpha)$ belongs to E_1 . We apply our first Basic Lemma 7.7.4 and determine a value-dictating subsread D of C_n , such that for all α in D both ${}^s\alpha$ and ${}^t(f|\alpha)$ belong to E_1 and for all u , if $u \perp s$ and u is almost completely free in C_n , then u is almost completely free in D . We then define: $\gamma(n) := 0$ and $\delta(n) := 0$ and let C_{n+1} be the set of all α in D passing through $\overline{\text{min}(D)}(n+1)$.

- (***) s is a non-final position in σ .

We then know: for all α in C_n , if ${}^s\alpha$ belongs to $A_{(s\sigma)}$, then ${}^t(f|\alpha)$ belongs to $E_{(t\sigma)} = E_{(s\sigma)}$, and ${}^s\sigma$ is a non-zero hereditarily repetitive stump different from 1^* . We apply our Main Lemma 7.7.7 and determine m, p and a value dictating subsread D of C_n such that ${}^{s* \langle m \rangle} \sigma = {}^{t* \langle p \rangle} \sigma$ and $s * \langle m \rangle$ is completely free in D , and for all α in D , if ${}^{s* \langle m \rangle} \alpha$ belongs to $E_{(s* \langle m \rangle \sigma)}$, then ${}^{t* \langle p \rangle} (f|\alpha)$ belongs to $A_{(t* \langle p \rangle \sigma)}$ and for every u , if $u \perp s$ and u is almost completely free in C_n , then u is almost completely free in D . We now distinguish two further subcases.

- (***)' $s * \langle m \rangle$ and therefore also $t * \langle p \rangle$ are final positions in σ . We now define $\gamma(n) := m$ and $\delta(n) := p$ and we let D' be the set of all α in D such that ${}^{s* \langle m \rangle} \alpha$ belongs to A_1 . We let C_{n+1} be the set of all α in D' passing through $\overline{\text{Min}(D')}(n+1)$.

Observe that according to the Second Basic Lemma 7.7.5 for all α in C_{n+1} , both ${}^{s* \langle m \rangle} \alpha$ and ${}^{t* \langle p \rangle} (f|\alpha)$ belong to A_1 .

- (***)'' $s * \langle m \rangle$ and therefore also $t * \langle p \rangle$ are non-final positions in σ . We now define

$\gamma(n) := m$ and $\delta(n) := p$ and we let C_{n+1} be the set of all α in D passing through $\overline{\text{Min}(D)}(n+1)$.

Observe that for all α in C_{n+1} , for all j , if $^{s*}\langle m, j \rangle \alpha$ belongs to $A_{(^{s*}\langle m, j \rangle \sigma)}$ then $^{t*}\langle p, j \rangle (f|\alpha)$ belongs to $E_{(^{t*}\langle p, j \rangle \sigma)}$.

This completes the description of the construction. Let α be the unique element of \mathcal{N} that belongs to every C_n . Observe that for every final position s in σ I-obeying γ , if $\text{length}(s)$ is even, then $^s\alpha$ belongs to E_1 and if $\text{length}(s)$ is odd, then $^s\alpha$ belongs to A_1 . Therefore α belongs to E_σ . Observe that for every final position t in σ I-obeying δ , if $\text{length}(t)$ is even, then $^t(f|\alpha)$ belongs to E_1 and if $\text{length}(t)$ is odd, then $^t(f|\alpha)$ belongs to A_1 . Therefore $f|\alpha$ belongs to E_σ . \square

7.7.10 Theorem: (Intuitionistic Borel Hierarchy Theorem, Second Part)

Let σ be a non-zero hereditarily repetitive stump and let f be a function from \mathcal{N} to \mathcal{N} mapping E_σ into A_σ .

There exists α in \mathcal{N} such that both α itself and $f|\alpha$ belong to A_σ .

Proof: We know from Theorem 7.4.1(ii) that the class $\mathbf{\Pi}_\sigma^0$ is closed under the operation of countable union and construct a function h from \mathcal{N} to \mathcal{N} such that for every α , $h|\alpha$ belongs to A_σ if and only if, for each n , α^n belongs to A_σ . Let τ be the non-empty stump such that, for each n , τ^n equals σ . τ is sometimes called the *successor* of σ . We let g be a function from \mathcal{N} to \mathcal{N} such that, for each α , for each n , $(g|\alpha)^n = f|(\alpha^n)$. Finally, we let k be a function from \mathcal{N} to \mathcal{N} such that, for each α , for each n , $(k|\alpha)^n = h|(g|\alpha)$. Observe that, for each α , if α belongs to A_τ , then for each n , α^n belongs to E_σ , and $(g|\alpha)^n$ belongs to A_σ , therefore $h|(g|\alpha)$ belongs to A_σ and $k|\alpha$ belongs to E_τ . Using Theorem 7.7.9 we find β such that both β itself and $k|\beta$ belong to E_τ . Now find n such that β^n belongs to A_σ , and observe: $h|(g|\beta)$ belongs to A_σ , therefore $(g|\beta)^n = f|(\beta^n)$ belongs to A_σ . \square

8 The never-ending productivity of disjunction

We show that, for every non-zero hereditarily repetitive stump σ , the set $D(A_1, A_\sigma)$ does not reduce to the set $A_{S(\sigma)}$ and that, for each n , the set $D^{n+1}(A_\sigma)$ does not reduce to the set $D^n(A_\sigma)$. We thus answer a question asked but not answered in [17]. A special case of this result has been shown in [24].

8.1 Let T be the set $\{\underline{0}\} \cup \{\underline{0}n * \langle 1 \rangle * \underline{0} | n \in \mathbb{N}\}$. We studied this countable set in Section 5.9. We now prove a simple fact that we want to use in establishing the main result of this Section.

8.2 Lemma: For all subsets A, B of the closure \overline{T} of T , if \overline{T} forms part of $A \cup B$, then there exists n such that either for all p , if $p > n$, then $\overline{0}p * \langle 1 \rangle * \underline{0}$ belongs to A , or for all p , if $p > n$, then $\overline{0}p * \langle 1 \rangle * \underline{0}$ belongs to B .

Proof: Applying the Continuity Principle we find n such that either every α in \overline{T} passing through $\overline{0}n$ belongs to A , or every α in \overline{T} passing through $\overline{0}n$ belongs to B . The conclusion follows easily. \square

8.3 Lemma: Let σ be a non-zero hereditarily repetitive stump.

- (i) Let X be a subset of \mathcal{N} reducing to E_σ , and let f be a function from \mathcal{N} to \mathcal{N} mapping A_σ into X .
There exists α in \mathcal{N} such that α belongs to E_σ and $f|\alpha$ belongs to X .
- (ii) For every n , for every n -sequence $\tau_0, \dots, \tau_{n-1}$ of non-zero hereditarily repetitive stumps, if for each $i < n$, $\tau_i < \sigma$, then $C_{i=0}^{n-1}(A_{\tau_i})$ reduces to E_σ .

Proof: (i) Let g be a function from \mathcal{N} to \mathcal{N} reducing X to E_σ . Let h be the function from \mathcal{N} to \mathcal{N} such that, for every α , $h|\alpha = g|(f|\alpha)$. Observe that h maps A_σ into E_σ , and using Theorem 7.7.9, find α such that both α and $h|\alpha$ belong to E_σ , therefore $f|\alpha$ belongs to X .

(ii) Remark that each set A_{τ_i} reduces to E_σ and that the set $C^2(E_\sigma)$ reduces to E_σ as the class Σ_σ^0 is closed under the operation of intersection of sets. \square

8.4 Lemma:

Let σ be a non-zero hereditarily repetitive stump.

Let α be an element of A_σ and let c be a natural number such that for every t II-obeying c , if t belongs to σ and $\text{length}(t)$ is even, then ${}^t\alpha$ belongs to $A_{({}^t\sigma)}$.

Then there exists γ passing through c such that α coincides with $\text{Corr}_{II}^\sigma(\gamma, \alpha)$.

Proof: Let H be the set of all numbers t of even length that belong to σ and II-obey c . Observe that H is a finite set and calculate k such that $2k = \max\{\text{length}(t) | t \in H\}$. We wish to prove, for every $i \leq k$, for every t in H , if $\text{length}(t) = 2k - 2i$, then there exists γ passing through c such that ${}^t\alpha$ coincides with ${}^t(\text{Corr}_{II}^\sigma(\gamma, \alpha))$. We use induction. Assume that t belongs to H , and, for every s in H such that $\text{length}(s) > \text{length}(t)$, there exists γ such that ${}^s\alpha$ coincides with ${}^s(\text{Corr}_{II}^\sigma(\gamma, \alpha))$. Calculate j such that $\text{length}(t) = j$ and find m such that $t = \langle (m)_0, c(\langle (m)_0 \rangle), (m)_1, c(\langle (m)_0, (m)_1 \rangle), \dots, (m)_{j-1}, c(m) \rangle$. We list the finitely many elements of H such that $\text{length}(u) = j + 2$, calling them $u_0, \dots, u_{\ell-1}$. We may assume that for each $p < \ell$, $u_p = t * \langle p, c(m * \langle p \rangle) \rangle$. For each $p < \ell$ we determine γ_p such that ${}^{(u_p)}(\text{Corr}_{II}^\sigma(\gamma_p, \alpha)) = {}^{(u_p)}\alpha$. We also determine δ such that ${}^t(\text{Corr}_{II}^\sigma(\delta, \alpha)) = {}^t\alpha$. We then define γ passing through c such that, for each p , if $p < \ell$, then ${}^{m*\langle p \rangle}\gamma = {}^{m*\langle p \rangle}(\gamma_p)$ and, if $p \geq \ell$, then ${}^{m*\langle p \rangle}\gamma = {}^{m*\langle p \rangle}\delta$, and observe that ${}^t(\text{Corr}_{II}^\sigma(\gamma, \alpha))$ coincides with ${}^t\alpha$.

After k steps we obtain the conclusion that there exists γ passing through c such that α coincides with $\text{Corr}_{II}^\sigma(\gamma, \alpha)$. \square

8.5 Lemma:

Let σ be a non-zero hereditarily repetitive stump.

Let g be a function from \mathcal{N} to \mathcal{N} such that, for every α , $g|\alpha$ belongs to A_σ .

Then for every α there exists γ, ε such that $g|\alpha$ coincides with $\text{Corr}_{II}^\sigma(\gamma, g|\alpha)$

and for every n , for every β , if β passes through $\bar{\alpha}(\varepsilon(n))$, then there exists

δ passing through $\bar{\gamma}n$ such that $g|\beta$ coincides with $\text{Corr}_{II}^\sigma(\delta, g|\beta)$.

Proof: The conclusion of this Lemma follows easily if one applies the Second Axiom of Continuous Choice, but we want to make it clear that one needs only the Continuity Principle and the First Axiom of Dependent Choices. Observe that for every n , for every α , we may determine c such that $\text{length}(c) = n$ and there exists γ passing through c with the property that $g|\alpha$ coincides with $\text{Corr}_{II}^\sigma(\gamma, g|\alpha)$. Using the Continuity Principle, we see that for every n , for every α , there exists m, c such that $\text{length}(c) = n$ and for all β passing through $\bar{\alpha}m$ there exists δ passing through c with the property that $g|\beta$ coincides with $\text{Corr}_{II}^\sigma(\delta, g|\beta)$.

Now let α belong to \mathcal{N} . We consider the set X of all pairs $\langle m, c \rangle$ such that for every β passing through $\bar{\alpha}m$ there exists δ passing through c with the property that $g|\beta$ coincides with $\text{Corr}_{II}^\sigma(\delta, g|\beta)$.

We have just seen that X contains at least one member.

Another application of the Continuity Principle ensures that to any $\langle m, c \rangle$ in X one may find $\langle p, d \rangle$ in X such that $\text{length}(d) = \text{length}(c) + 1$ and c is an initial part of d . Applying the First Axiom of Dependent Choices we find β, ζ in \mathcal{N} such that for each p , $\text{length}(\zeta(n)) = n$, and $\zeta(n)$ is an initial part of $\zeta(n+1)$, and for each β , if β passes through $\bar{\alpha}(\varepsilon(n))$, then there exists δ passing through $\zeta(n)$ with the property that $g|\beta$ coincides with $\text{Corr}_{II}^\sigma(\delta, g|\beta)$. We now let γ be the element of \mathcal{N} such that for each n , $\bar{\gamma}n = \zeta(n)$. Observe that for every s , if s II-obey γ , then there exists n such that for every p , if $p > n$, then s II-obey $\zeta(p)$. It follows that $g|\alpha$ coincides with $\text{Corr}_{II}^\sigma(\gamma, g|\alpha)$. \square

8.6 Theorem:

Let σ be a non-zero hereditarily repetitive stump. The set $D(A_1, A_\sigma)$ does not reduce to the set $A_{S(\sigma)}$.

Proof: Assume that f is a function from \mathcal{N} to \mathcal{N} reducing $D(A_1, A_\sigma)$ to $A_{S(\sigma)}$. We want to obtain a contradiction.

We define a function h from \mathcal{N} to \mathcal{N} such that, for every α , $(h|\alpha)^1$ coincides with $\text{Corr}_{II}^\sigma(\alpha^1, \alpha^{0,1})$ and, for every t , if there is no j such that $t = \langle 1 \rangle * j$, then $(h|\alpha)(t) := \alpha^0(t)$.

Observe that, for each α , the sequence $(h|\alpha)^1$ belongs to A_σ and the sequence $f|(h|\alpha)$ belongs to $A_{S(\sigma)}$. Observe that $(h|\underline{0})^0$ belongs to A_1 and that $(h|\underline{0})^1$ belongs to A_σ . We apply Lemma 8.5 and determine γ, ε such that the sequence $f|(h|\underline{0})$ coincides with $Corr_{II}^{S(\sigma)}(\gamma, f|(h|\underline{0}))$ and for every n , for every α , if α passes through $\overline{0}(\varepsilon(n))$, then there exists δ passing through $\overline{\gamma}n$ such that $f|(h|\alpha)$ coincides with $Corr_{II}^{S(\sigma)}(\delta, f|\alpha)$. We assume that ε is strictly increasing, that is, for each n , $\varepsilon(n) < \varepsilon(n+1)$. We now want to define elements β, δ of \mathcal{N} with the following properties.

- (i) for each n , $\beta^{2n,1}$ belongs to E_σ and $\beta^{2n+1,0}$ belongs to E_1 .
- (ii) for each n , β^n passes through $(h|\underline{0})(\varepsilon(n))$ and the sequence δ^n passes through $\overline{\gamma}n$ and $f|\beta^n$ coincides with $Corr_{II}^{S(\sigma)}(\delta^n, f|\beta^n)$.

Let n belong to \mathbb{N} . We show how to define β^{2n} and β^{2n+1} , and also δ^{2n} and δ^{2n+1} . We let C be the set of all α such that α passes through $(h|\underline{0})(\varepsilon(2n))$ and, for all j , if $j < \varepsilon(2n)$ then $\alpha^{1,j} = (h|\underline{0})^{1,j}$. Let X be the set of α in C such that for each $j \geq \varepsilon(2n)$, $\alpha^{1,j}$ belongs to $E_{(\sigma^j)}$. Observe that for each α , if α belongs to X , then there exists ζ passing through $\overline{0}\varepsilon(2n)$ such that $h|\zeta = \alpha$, therefore there exists δ passing through $\overline{\gamma}(2n)$ such that $f|\alpha$ coincides with $Corr_{II}^{S(\sigma)}(\delta, f|\alpha)$, in particular, for every t of even length belonging to $S(\sigma)$ and II-obeying $\overline{\gamma}(2n)$ the sequence ${}^t(f|\alpha)$ belongs to $A_{({}^t(S(\sigma)))}$. We now define a function g from \mathcal{N} to \mathcal{N} mapping A_σ into X .

We first construct μ in \mathbb{N} such that μ is strictly increasing, $\varepsilon(2n) \leq \mu(0)$ and for each i , σ^i equals $\sigma^{\mu(i)}$. We then define: for every α , for every i , $(g|\alpha)^{1,\mu(i)} := \alpha^i$ and for each p , if there do not exist i, s such that $p = \langle 1, \mu(i) \rangle * s$, then $(g|\alpha)(p) := (h|\underline{0})(p)$. We now consider the function k from \mathcal{N} to \mathcal{N} such that for every α , $k|\alpha = f|(g|\alpha)$. Observe that for every α , if α belongs to A_σ , then $(g|\alpha)^1$ belongs to A_σ and for every t of positive even length belonging to σ and II-obeying $\overline{\gamma}(2n)$, the sequence ${}^t(k|\alpha)$ belongs to $A_{({}^t(S(\sigma)))}$ and for every such t , ${}^t(S(\sigma)) < \sigma$.

The set Y consisting of all ζ such that for every such t , ${}^t\zeta$ belongs to $A_{({}^t(S(\sigma)))}$ reduces to E_σ . We apply Lemma 8.3 and find α in E_σ such that $k|\alpha$ belongs to Y . We define $\beta^{2n} := g|\alpha$. Observe that $\beta^{2n,1}$ belongs to E_σ , as α does so. Observe that $\beta^{2n,0}$ belongs to A_1 , therefore $f|\beta^{2n,0}$ belongs to A_σ . Also α belongs to Y , that is, for every t of positive even length belonging to σ and II-obeying $\overline{\gamma}n$, the sequence ${}^t(f|\beta^{2n,0}) = {}^t(k|\alpha)$ belongs to $A_{({}^t(S(\sigma)))}$. Applying Lemma 8.4, we may define δ^{2n} passing through $\overline{\gamma}(2n)$ such that $f|(\beta^{2n,0})$ coincides with $Corr_{II}^{S(\sigma)}(\delta^{2n}, f|(\beta^{2n,0}))$.

This completes the definition of β^{2n} and δ^{2n} .

We also define: $\beta^{2n+1} := h|(\overline{0}(\varepsilon(2n+1)) * \underline{1})$. Observe that $\beta^{2n+1,0}$ belongs to E_1 and $\beta^{2n+1,1}$ belongs to A_σ , and $f|(\beta^{2n+1})$ belongs to $A_{S(\sigma)}$ and we may define δ^{2n+1} passing through $\overline{\gamma}(2n+1)$ such that $f|(\beta^{2n+1})$ coincides with $Corr_{II}^{S(\sigma)}(\delta^{2n+1}, f|(\beta^{2n+1}))$. Let T be the set $\{\underline{0}\} \cup \{\overline{0}n * \langle 1 \rangle * \underline{0} | n \in \mathbb{N}\}$.

We build functions b and d from the closure \overline{T} of T to \mathcal{N} such that for every n ,

$b|(\bar{0}n * \langle 1 \rangle * \underline{0})$ equals β^n and $d|(\bar{0}n * \langle 1 \rangle * \underline{0})$ equals δ^n . Observe that $b|\underline{0}$ coincides with $h|\underline{0}$ and $d|\underline{0}$ coincides with γ , therefore $f|(b|\underline{0})$ coincides with $Corr_{II}^{S(\sigma)}(d|\underline{0}, f|(b|\underline{0}))$. We claim that for every α in \bar{T} , the sequence $f|(b|\alpha)$ coincides with the sequence $Corr_{II}^{S(\sigma)}(d|\alpha, f|(b|\alpha))$.

For suppose that for some α, p , the value of the sequence $f|(b|\alpha)$ at p differs from the value of $Corr_{II}^{S(\sigma)}(d|\alpha, f|(b|\alpha))$ at p . Then the sequence α must be different from every sequence $\bar{0}n * \langle 1 \rangle * \underline{0}$, therefore $\alpha = \underline{0}$. But the sequence $b|\underline{0}$ coincides with the sequence $Corr_{II}^{S(\sigma)}(d|\underline{0}, f|(b|\underline{0}))$. Contradiction.

We conclude that for every α in T , $f|(b|\alpha)$ belongs to $A_{S(\sigma)}$, therefore $b|\alpha$ belongs to $D(A_1, A_\sigma)$.

Applying Lemma 8.1 we find n such that either for every $p > n$ if $p > n$, then $b|(\bar{0}p * \langle 1 \rangle * \underline{0}) = \beta^{p,0}$ belongs to A_1 , or for every p , if $p > n$, then $\beta^{p,1}$ belongs to A_σ . Contradiction. \square

8.7 Theorem: Let σ be a non-zero hereditarily repetitive stump.

For each positive n , the set $D(A_1, D^n(A_\sigma))$ does not reduce to the set $D^n(A_{S(\sigma)})$.

Proof: Assume that n is a positive natural number and f is a function from \mathcal{N} to \mathcal{N} reducing $D(A_1, D^n(A_\sigma))$ to $D^n(A_{S(\sigma)})$. We let X be the set of all α such that $\alpha^0 = \underline{0}$ belongs to A_1 , and for each $i < n$, we let h_i be the function from \mathcal{N} to \mathcal{N} such that for every α , the sequence $(h_i|\alpha)^{1,i}$ coincides with the sequence $Corr_{II}^\sigma(\alpha^1, \alpha^{0,1,i})$ and for every t , if there is no j such that $t = \langle 1, i \rangle * j$, then $(h_i|\alpha)(t) = \alpha^0(t)$. Applying the Continuity Principle we first calculate p, q such that $q < n$ and, for every α in X , if α passes through $\bar{0}p$, then $(f|\alpha)^q$ belongs to $A_{S(\sigma)}$. Applying the Continuity Principle again, we find for each $i < n$ numbers p_i, q_i such that for every α , if α passes through $\bar{0}p_i$ then $(f|(h_i|\alpha))^{q_i}$ belongs to A . We now distinguish two cases. *First Case.* We find i such that $q_i = q$. We then consider the set Z consisting of all α such that α passes through $\bar{0}p$ and through $\bar{0}p_i$ and for each $j < p_i$, $\alpha^{1,i,j}$ coincides with $(h_i|\underline{0})^{1,i,j}$. Observe that for every α in Z , if $\alpha^{1,i}$ belongs to A_σ , there exists β passing through $\bar{0}p_i$ such that $\alpha = h_i|\beta$ and therefore $(f|\alpha)^q$ belongs to $A_{S(\sigma)}$. Also if α^0 belongs to A_1 , $(f|\alpha)^q$ belongs to $A_{S(\sigma)}$. We leave it to the reader to conclude from this that the set $D(A_1, A_\sigma)$ reduces to the set $A_{S(\sigma)}$. We now have a contradiction, according to Theorem 8.6. *Second Case.* We find $i, j < n$ such that $i \neq j$ and $q_i = q_j$. We now let p be the greatest of p_0, p_1 , and let Z be the set of all α passing through $\bar{0}p$ such that for every $k < p$, both $\alpha^{1,i,k}$ coincides with $(h_i|\underline{0})^{1,i,k}$ and $\alpha^{1,j,k}$ coincides with $(h_j|\underline{0})^{1,j,k}$. Observe that for every α in Z , if $\alpha^{1,i}$ belongs to A_σ or $\alpha^{1,j}$ belongs to A_σ , $(f|\alpha)^{q_i}$ belongs to A_σ . We may conclude from this that the set $D(A_\sigma, A_\sigma)$ reduces to the set $A_{S(\sigma)}$, therefore $D(A_1, A_\sigma)$ reduces to $A_{S(\sigma)}$, and this contradicts Theorem 8.6. \square

8.8 Corollary: Let σ be a non-zero hereditarily repetitive stump.

For each positive n , the set $D^{n+1}(A_\sigma)$ does not reduce to the set $D^n(A_\sigma)$.

Proof: The statement easily follows from Theorem 8.7. \square

9 On (strictly) analytic, co-analytic and projective sets.

Using Brouwer's Thesis, we recover some famous results on strictly analytic sets. We introduce the wider class of analytic sets and its counterpart, the class of co-analytic sets. We show the collapse of the projective hierarchy.

9.1 Let X be a subset of \mathcal{N} . X is called *strictly analytic* if and only if there exists a function γ from \mathcal{N} to \mathcal{N} such that X coincides with the range of γ , that is, for every α , α belongs to X if and only if, for some β , α coincides with $\gamma|\beta$. Every strictly analytic subset X of \mathcal{N} is *inhabited*, that is, there exists α such that α belongs to X .

Every one of the leading sets of the Borel hierarchy, that is, every set A_σ and every set E_σ , is strictly analytic.

9.2 Theorem: (*Lusin Separation Theorem*)

Let γ, δ be functions from \mathcal{N} to \mathcal{N} such that for every α, β , $\gamma|\alpha$ is apart from $\delta|\beta$. There exist positively Borel sets C, D , such that for every α, β , $\gamma|\alpha$ belongs to C and $\delta|\beta$ belongs to D , and every element of C is apart from every element of D .

Proof: We use the Principle Induction on Monotone Bars, 1.6.4. We define subsets P, Q of $\mathbb{N} \times \mathbb{N}$, as follows: for every a, b in \mathbb{N} ,

- $P(a, b) :=$ For every α passing through a , for every β passing through b , the sequence $\gamma|\alpha$ is apart from the sequence $\delta|\beta$.
- $Q(a, b) :=$ There exist positively Borel sets C, D such that for every α passing through a , for every β passing through b , $\gamma|\alpha$ belongs to C and $\delta|\beta$ belongs to D , and every element of C is apart from every element of D .

It will be clear that P is a monotone bar in $\mathcal{N} \times \mathcal{N}$ and that P is a subset of Q . Now assume a, b belong to \mathbb{N} and for every m , for every n , the pair $(a * \langle m \rangle, b * \langle n \rangle)$ belongs to Q . Using the Second Axiom of Countable Choice, we determine, for every m, n , positively Borel sets $C_{m,n}$ and $D_{m,n}$ such that for every α passing through $a * \langle m \rangle$, for every β passing through $b * \langle n \rangle$, $\gamma|\alpha$ belongs to $C_{m,n}$ and $\delta|\beta$ belongs to $D_{m,n}$ and every element of $C_{m,n}$ is apart from every element of $D_{m,n}$. Define $C := \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} C_{m,n}$ and $D := \bigcap_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} D_{m,n}$ and remark that for every α passing through a , for every β

passing through b , $\gamma|\alpha$ belongs to C and $\delta|\beta$ belongs to D , and every element of C is apart from every element of D .

The Principle of Double Induction on Monotone Bars allows us to conclude that the pair $(\langle \rangle, \langle \rangle)$ belongs to Q , and the conclusion of the Theorem follows. \square

9.3 Theorem: (*One half of Lusin's Regular Representation Theorem*)

Let γ be a function from \mathcal{N} to \mathcal{N} , such that for all α, β , if α is apart from β , then $\gamma|\alpha$ is apart from $\gamma|\beta$.

Then the range of γ , that is, the set of all α coinciding with some $\gamma|\beta$, is positively Borel.

Proof: Assume that γ is a function from \mathcal{N} to \mathcal{N} and that for all α, β , if α is apart from β , then $\gamma|\alpha$ is apart from $\gamma|\beta$.

We intend to define a sequence H_0, H_1, \dots of positively Borel sets such that the range of γ coincides with $\bigcap_{n \in \mathbb{N}} H_n$.

To this end, we first define a sequence B_0, B_1, \dots of decidable subsets of \mathbb{N} . Each of these will be a bar in \mathcal{N} .

We let B_0 be the set $\{\langle \rangle\}$ consisting of the empty sequence only. For each n , we let B_{n+1} be the set of all a of minimal length such that some proper initial segment of a belongs to B_n , and for some initial part b of a , $\gamma^n(b) \neq 0$. Observe that for each n , for all a in B_n , for all α, β passing through a , $(\gamma|\alpha)n = (\gamma|\beta)n$. We let ε be an element of \mathcal{N} such that for each n , for each a in B_n , for each α passing through a , $(\gamma|\alpha)n$ equals $\varepsilon^n(a)$. Using Theorem 9.2 and the Second Axiom of Countable Choice we find for each n , for all a, b in B_n such that $a \neq b$, positively Borel sets $C_{a,b}, D_{a,b}$ such that for all α , if α passes through a , then $\gamma|\alpha$ belongs to $C_{a,b}$ and, if α passes through b , then $\gamma|\alpha$ belongs to $D_{a,b}$, and every element of $C_{a,b}$ is apart from every element of $D_{a,b}$. For each n , for each a in B_n we let K_a be the set of all β passing through $\varepsilon^n(a)$ and belonging to every set $C_{a,b} \cap D_{b,a}$, where b belongs to B_n but differs from a . Remark that for all n , for all a, b in B_n , if $a \neq b$, then every element of K_a is apart from every element of K_b . For each n , for each a in B_n we let L_a be the set of all β belonging to every set K_b , where b is an initial segment of a , and for some $i \leq n$, b belongs to B_i . We define, for each n , $H_n := \bigcup_{a \in B_n} L_a$.

It will be clear that the range of γ forms part of every set H_n . Now assume that β belongs to every set H_n . For each n , we determine a_n in B_n such that β belongs to $L_{(a_n)}$. Observe that for each n , a_n must be a proper initial segment of a_{n+1} and consider the sequence α that passes through every a_n .

Observe that for each n , β belongs to $K_{(a_n)}$ and therefore passes through $(\gamma|\alpha)n$, so β coincides with $\gamma|\alpha$. \square

9.4 Recall from Section 1.3.3, that a subset X of \mathcal{N} is weakly closed if and only if every α such that, for each n , $\bar{\alpha}n$ contains an element of X belongs to X . If in

addition, one may decide for each a , if a contains an element of X or not, then X is a spread. As subset X of \mathcal{N} is closed if and only if there exists a decidable subset C of \mathbb{N} such that, for every α , α belongs to X if and only if, for each n , $\bar{\alpha}n$ belongs to C .

9.5 Theorem:

- (i) Every spread is a strictly analytic subset of \mathcal{N} .
- (ii) For every weakly-closed subset of X of \mathcal{N} , X is strictly analytic if and only if there exists δ such that for every a , a contains a member of X if and only if, for some n , $\delta(n)$ equals a .
- (iii) Not every strictly analytic and closed subset of X is a spread.
- (iv) Not every inhabited and closed subset of X is strictly analytic.

Proof: (i) For every inhabited spread X may define a so-called retraction from \mathcal{N} to X , that is a function γ from \mathcal{N} to X such that for every α in X , $\gamma|\alpha = \alpha$. We have done so in Section 1.3.5. We conclude that every spread is a strictly analytic subset of \mathcal{N} .

(ii) Let us first assume that γ is a function from \mathcal{N} to \mathcal{N} . Observe that for each a , a contains a member of the range of γ if and only if there exists b such that for each $i < \text{length}(a)$ there exists an initial segment c of b such that $\gamma^i(c) = (a)_i + 1$ and for all initial segments d of c , $\gamma^i(d) = 0$.

It is easy to construct δ enumerating all numbers a with this property. Now assume that we have a δ enumerating all numbers a containing an element of X . Observe that for every n there exist m, p such that $\delta(m) = \delta(n) * \langle p \rangle$. We construct ε such that for each a , $\text{length}(a) = \text{length}(\varepsilon(a))$, as follows. We define $\varepsilon(\langle \rangle) := \langle \rangle$ and for each a, n , $\varepsilon(a * \langle n \rangle) := \delta(n)$ if $\delta(n)$ is an immediate successor of $\varepsilon(a)$, and $\varepsilon(a * \langle n \rangle) = \varepsilon(a) * \langle p_0 \rangle$ where p_0 is such that for some k , $\delta(k) = \varepsilon(a) * \langle p_0 \rangle$ and for no $i < k$ there exists p such that $\delta(k) = \varepsilon(a) * \langle p \rangle$. We then let γ be a function from \mathcal{N} to \mathcal{N} such that for every n , for every α , $\gamma|\alpha$ passes through $\varepsilon(\bar{\alpha}n)$. X will coincide with the range of γ .

(iii) For every α , we define a decidable subset C_α of \mathbb{N} , as follows. For each a , a belongs to C_α if and only if either $\underline{0}$ passes through a , or $\text{length}(a) > 0$ and $(a)_0 = 1$ and, if $\text{length}(a) > 1$, there exists $n < (a)_1$ such that $\alpha(n) \neq 0$. We let X_α be the set of all β such that, for each n , $\bar{\beta}n$ belongs to C_α . Observe that X_α is closed and inhabited. We claim that X_α is also strictly analytical, as, for every α , we may construct δ enumerating the natural numbers containing an element of X_α as follows. For each a , if either $\underline{0}$ passes through a or $\text{length}(a) > 1$ and $(a)_0 = 1$ and there exists $n < (a)_1$ such that $\alpha(n) \neq 0$, then $\delta(a) := a$, and if $\text{length}(a) = 1$ and $(a)_0$ is the least n such that $\alpha(n) \neq 0$, then $\delta(a) := \langle 1 \rangle$, and if these conditions do not apply then $\delta(a) := \langle \rangle$. Assume that for each α , X_α is a spread, therefore $\langle 1 \rangle$ contains an element of X_α or not, therefore *either*: for some n , $\alpha(n) \neq 0$, *or*: $\alpha = \underline{0}$. We easily obtain a contradiction by the Continuity Principle.

(iv) For each α , we let Y_α be the set of all β such that, for each n , either $\bar{\beta}n = \underline{0}n$

or both $\bar{\beta}n = \bar{1}n$ and $\bar{\alpha}n = \bar{0}n$. Observe that, for each α , Y_α is closed. Assume that, for each α , Y_α is strictly analytical, that is, there exists δ enumerating the set of all numbers a containing an element of Y_α . Applying the Second Axiom of Continuous Choice we find a function e from \mathcal{N} to \mathcal{N} such that for all α , $e|\alpha$ enumerates the numbers containing an element of Y_α . Calculate m such that $(e|\underline{0})(m) = \langle 1 \rangle$. Calculate p such that for every α , if $\bar{\alpha}p = \bar{0}p$, then $(e|\alpha)(m) = (e|\underline{0})(m) = \langle 1 \rangle$, therefore $\underline{1}$ belongs to Y_α and $\alpha = \underline{0}$. Contradiction. \square

9.6 Suppose that γ belongs to **Fun**, that is, for every α there exists n such that $\gamma(\bar{\alpha}n) \neq 0$. Let a be a natural number. We say that a *belongs to* γ if and only if for every initial segment b of a , $\gamma(b) = 0$. Observe that **Stp** is a subset of **Fun**, and that our terminology accords with the one introduced in Section 1.5.3. Suppose that γ, δ belong to **Fun**. Let ε be an element of \mathcal{N} . We say that ε *embeds* γ *into* δ if and only if ε maps every number belonging to γ onto a number belonging to δ in such a way that for every a, b belonging to γ , if a is a proper initial segment of b , then $\varepsilon(a)$ is a proper initial segment of $\varepsilon(b)$.

We say that γ *embeds into* δ , notation: $\gamma \leq^* \delta$, if and only if there exists ε embedding γ into δ . We say that γ *properly embeds into* δ , notation: $\gamma <^* \delta$, if and only if, for some n , γ embeds into δ^n .

Let γ belong to **Fun**. We define an element of \mathcal{N} , called *the successor of* γ , notation: $S(\gamma)$, as follows. $(S(\gamma))(\langle \rangle) := 0$ and, for each n , $(S(\gamma))^n := \gamma$.

9.7 Theorem:

- (i) For all α in **Fun**, $\alpha \leq^* \alpha$ and not: $\alpha <^* \alpha$.
 For all α, β in **Fun**, if $\alpha <^* \beta$, then $\alpha \leq^* \beta$.
 For all α, β, γ in **Fun**, if $\alpha \leq^* \beta$ and $\beta <^* \gamma$, then $\alpha <^* \gamma$, and: if $\alpha <^* \beta$ and $\beta \leq^* \gamma$, then $\alpha <^* \gamma$, and if $\alpha \leq^* \beta$ and $\beta \leq^* \gamma$, then $\alpha \leq^* \gamma$.
 For all α, β in **Fun**, $\alpha \leq^* \beta$ if and only if $\alpha <^* S(\beta)$.
- (ii) (*Brouwer's Thesis*) For every γ in **Fun** there exists a stump σ such that the set of all natural numbers belonging to γ coincides with the set of all numbers belonging to σ .
- (iii) (*Cantor's Argument*) For every function γ from \mathcal{N} to **Fun** there exists α in **Fun** such that, for every β , $\gamma|\beta$ is apart from α .
- (iv) (*Boundedness Theorem*) For every function γ from \mathcal{N} to **Fun** there exists α in **Fun** such that, for every β , $\gamma|\beta$ embeds into α .

Proof: We leave the proofs of (i) and (ii) to the reader. As to (iii), suppose γ is a function from \mathcal{N} to **Fun**, and let α be an element of **Fun** such that, for every β , $\alpha(\beta) = (\gamma|\beta)(\beta) + 1$. Clearly, α is apart from every $\gamma|\beta$. We now prove (iv). Again suppose γ is a function from \mathcal{N} to **Fun**. Observe that for every β , for every δ , there exists n such that $(\gamma|\beta)(\bar{\delta}n) \neq 0$ and there exists m such that $\gamma^{\bar{\delta}n}(\bar{\beta}m) = (\gamma|\beta)(\bar{\delta}n) + 1$.

We define α in \mathcal{C} such that for every a, n , if $n = \text{length}(a)$ and $a = \langle (a)_0, \dots, (a)_{n-1} \rangle$, then $\alpha(a) = 1$ if and only if there exist i, j such that $2i < n$ and $2j + 1 < n$ and $\gamma^{\langle (a)_0, (a)_2, \dots, (a)_{2i} \rangle}(\langle (a)_1, (a)_3, \dots, (a)_{2j+1} \rangle)$ differs from 0. Clearly, α belongs to **Fun**. Now consider any β in \mathcal{N} . Define an element ε of \mathcal{N} , as follows. For every d, n , if $n = \text{length}(d)$ and $d = \langle (d)_0, \dots, (d)_{n-1} \rangle$, then $\varepsilon(d) := \langle (d)_0, \beta(0), \dots, (d)_{n-1}, \beta(n-1) \rangle$. One verifies easily that ε embeds $\gamma|\beta$ into α . \boxtimes

9.8 We introduce two projection operations on the class of subsets of \mathcal{N} . For every subset X of \mathcal{N} , we let the *(existential) projection* of X , notation $Ex(X)$, be the set of all α such that, for some β , $\langle \alpha, \beta \rangle$ belongs to X .

For every subset X of \mathcal{N} , we let the *universal projection* of X , notation $Un(X)$, be the set of all α such that for every β , $\langle \alpha, \beta \rangle$ belongs to X .

A subset of \mathcal{N} is called *analytic* if and only if there exists a closed subset Y of \mathcal{N} such that X coincides with $Ex(Y)$. The class of the analytic subsets of \mathcal{N} is denoted by Σ_1^1 .

A subset X of \mathcal{N} is called *co-analytic* if and only if there exists an open subset Y of \mathcal{N} such that X coincides with $Un(Y)$. The class of the co-analytic subsets of \mathcal{N} is denoted by Π_1^1 .

We use S^* to denote the successor-function from \mathbb{N} to \mathbb{N} , and \circ to denote the operation of composition on \mathcal{N} . So, for each α, β, n , $\alpha \circ \beta(n) = \alpha(\beta(n))$.

9.9 Theorem: (*Closure Properties of the classes Σ_1^1 and Π_1^1*)

- (i) For every sequence X_0, X_1, \dots of subsets of \mathcal{N} , if, for each n , X_n belongs to Σ_1^1 , then both $\bigcup_{n \in \mathbb{N}} X_n$ and $\bigcap_{n \in \mathbb{N}} X_n$ belong to Σ_1^1 .
- (ii) For every subset X of \mathcal{N} , if X is positively Borel, then X is analytic.
- (iii) Every strictly analytic subset of \mathcal{N} is inhabited and analytic, but not every inhabited and closed subset of \mathcal{N} is strictly analytic.
- (iv) Every co-analytic subset of \mathcal{N} is perhapsive, and the set $D^2(A_1)$ is not co-analytic.
- (v) For every sequence X_0, X_1, \dots of subsets of \mathcal{N} , if, for each n , X_n belongs to Π_1^1 , then $\bigcap_{n \in \mathbb{N}} X_n$ belongs to Π_1^1 .
- (vi) Every Π_2^0 -subset of \mathcal{N} is co-analytic.
- (vii) For every subset X of \mathcal{N} , if X belongs to Σ_1^1 , then $Ex(X)$ belongs to Σ_1^1 and: if X belongs to Π_1^1 , then $Un(X)$ belongs to Π_1^1 .
- (viii) For all subsets X, Y of \mathcal{N} , if X reduces to Y and Y is analytic, then X is analytic, and: if X reduces to Y and Y is co-analytic, then X is co-analytic.
- (ix) For every subset X of \mathcal{N} , if X is strictly analytic, then $Ex(X)$ is strictly analytic. In particular, if X is a spread, or a weakly closed strictly analytic subset of \mathcal{N} , then $Ex(X)$ is strictly analytic.

Proof: (i) We use the Second Axiom of Countable Choice. Let Y_0, Y_1, \dots be a sequence of closed subsets of \mathcal{N} such that, for each n , $X_n = Ex(Y_n)$. We now define subsets Z_0, Z_1 of \mathcal{N} , as follows.

Z_0 is the set of all α such that $\langle \alpha^0, \alpha^1 \circ S^* \rangle$ belongs to $Y_{\alpha^1(0)}$ and Z_1 is the set of all α such that for all n , $\langle \alpha^0, \alpha^{1,n} \rangle$ belongs to Y_n . Observe that Z_0, Z_1 are closed, and $\bigcup_{n \in \mathbb{N}} X_n$ coincides with $Ex(Z_0)$ and $\bigcap_{n \in \mathbb{N}} X_n$ coincides with $Ex(Z_1)$.

(ii) follows easily from (i).

(iii) Let γ be a function from \mathcal{N} to \mathcal{N} . Let Z be the set of all α such that for each n, m , if $\gamma^n(\alpha^1 m) \neq 0$ and for every $i < m$, $\gamma^n(\alpha^1 i) = 0$, then $\gamma^n(\alpha^1 m) = \alpha^0(n) + 1$. Observe that Z is a closed subset of \mathcal{N} and that the range of γ coincides with $Ex(Z)$.

(iv) Let X be a co-analytical subset of \mathcal{N} and let Y be an open subset of \mathcal{N} such that $X = Un(Y)$. Now suppose we are given α, γ such that γ belongs to X , and: if α is apart from γ , then α belongs to X . We claim that α itself belongs to X . In order to see this, find a decidable subset C of \mathbb{N} such that for every δ , δ belongs to Y if and only if, for some n , $\bar{\delta}n$ belongs to C .

Let β be an element of \mathbb{N} . Find n such that $\overline{\langle \gamma, \beta \rangle}n$ belongs to C , and distinguish two cases. Either $\overline{\langle \alpha, \beta \rangle}n = \overline{\langle \gamma, \beta \rangle}n$, and $\overline{\langle \alpha, \beta \rangle}n$ belongs to C , and therefore $\langle \alpha, \beta \rangle$ belongs to Y or $\overline{\langle \alpha, \beta \rangle}n \neq \overline{\langle \gamma, \beta \rangle}n$, therefore α is apart from γ , and α belongs to X and $\langle \alpha, \beta \rangle$ belongs to Y . In any case $\langle \alpha, \beta \rangle$ belongs to Y . Therefore, for every β , $\langle \alpha, \beta \rangle$ belongs to Y , and α itself belongs to X . This shows that every co-analytical subset of \mathcal{N} is perhapsive.

The set $D^2(A_1)$ is not perhapsive, see Theorem 5.4, and therefore does not belong to Π_1^1 .

(v) We use the Second Axiom of Countable Choice. Let Y_0, Y_1, \dots be a sequence of open subsets of \mathcal{N} such that, for each n , $X_n = Un(Y_n)$. We define a subset Z of \mathcal{N} as follows. Z is the set of all α such that $\langle \alpha^0, \alpha^1 \circ S^* \rangle$ belongs to $Y_{\alpha^1(0)}$. Observe that Z is open and $\bigcap_{n \in \mathbb{N}} X_n$ coincides with $Un(Z)$.

(vi) Use (v) and the simple fact that every open subset of \mathcal{N} is co-analytical.

(vii) Suppose that X belongs to Σ_1^1 . Let Y be a closed subset of \mathcal{N} such that $X = Ex(Y)$. Observe that for every α , α belongs to $Ex(X)$ if and only if for some β , $\langle \alpha, \beta \rangle$ belongs to X if and only if for some β, γ , the sequence $\langle \langle \alpha, \beta \rangle, \gamma \rangle$ belongs to Y . Let Z be the set of all α such that $\langle \langle \alpha^0, \alpha^{1,0} \rangle, \alpha^{1,1} \rangle$ belongs to Y . Observe that Z is closed and $Ex(X)$ coincides with $Ex(Z)$, therefore $Ex(X)$ belongs to Σ_1^1 . The proof of the second statement is similar.

(viii) Let X, Y be subsets of \mathcal{N} and let γ be a function from \mathcal{N} to \mathcal{N} reducing X to Y . Assume that Y is analytical and let Z be a closed subset of \mathcal{N} such that Y coincides with $Ex(Z)$. Observe that for every α , α belongs to X if and only if $\gamma|\alpha$ belongs to Y if and only if for some β $\langle \gamma|\alpha, \beta \rangle$ belongs to Z .

Let V be the set of all α such that $\langle \gamma|\alpha^0, \alpha^1 \rangle$ belongs to Z . Observe that V is closed and X coincides with $Ex(V)$, therefore X is analytical. The proof of the second

statement is similar. \square

9.10 We have proved, in Theorem 4.7, that for every enumerable subset X of \mathcal{N} , the set $\text{Almost}^*(X)$ is perhapse. Observe that for such sets X , the set $\text{Almost}^*(X)$ is co-analytical, so Theorem 4.7(iii) follows from Theorem 9.9(iv).

We define subsets US_1^1 and UP_1^1 of \mathcal{N} as follows.

US_1^1 is the set of all α such that, for some β , the sequence $\langle \alpha^0, \langle \alpha^1, \beta \rangle \rangle$ belongs to UP_{1*} . UP_1^1 is the set of all α such that, for all β , the sequence $\langle \alpha^0, \langle \alpha^1, \beta \rangle \rangle$ belongs to US_{1*} . We call US_1^1, UP_1^1 the *cataloguing* sets of Σ_1^1, Π_1^1 , respectively.

We define subsets E_1^1 and A_1^1 of \mathcal{N} as follows.

E_1^1 is the set of all α such that for some β , for all n , $\alpha(\bar{\beta}n) = 0$. A_1^1 coincides with **Fun**, so A_1^1 consists of all α such that for every β there exists n such that $\alpha(\bar{\beta}n) \neq 0$. We call E_1^1, A_1^1 the *leading* sets of Σ_1^1, Π_1^1 , respectively.

9.11 Theorem: (*Classical remarks on cataloguing and leading sets in Σ_1^1 and Π_1^1*)

- (i) For every subset X of \mathcal{N} , X belongs to Σ_1^1 if and only if, for some α , X coincides with $US_1^1 \upharpoonright \alpha$, and: X belongs to Π_1^1 if and only if, for some α , X coincides with $UP_1^1 \upharpoonright \alpha$.
Moreover, UP_1^1 is the set of all α in \mathcal{N} apart from every β in US_1^1 .
- (ii) For every subset X of \mathcal{N} , X belongs to Σ_1^1 if and only if X reduces to E_1^1 , and: X belongs to Π_1^1 if and only if X reduces to A_1^1 .
Moreover, A_1^1 is the set of all α in \mathcal{N} apart from every β in E_1^1 .
- (iii) If UP_1^1 belongs to Σ_1^1 , and also if US_1^1 belongs to Π_1^1 , there exists α not belonging to either one of UP_1^1, US_1^1 .
- (iv) For every function f from \mathcal{N} to \mathcal{N} there exists α such that α belongs to A_1^1 if and only if $f| \alpha$ belongs to E_1^1 and also: α belongs to E_1^1 if and only if $f| \alpha$ belongs to E_1^1 .
- (v) If A_1^1 reduces to E_1^1 and also if E_1^1 reduces to A_1^1 , there exists α not belonging to either one of A_1^1, E_1^1 .

Proof: (i) Let X belong to Σ_1^1 and let Y be a closed subset of \mathcal{N} such that $X = \text{Ex}(Y)$. Find α such that Y coincides with $UP_{1*} \upharpoonright \alpha$ and observe that, for every β , β belongs to X if and only if, for some γ , $\langle \beta, \gamma \rangle$ belongs to Y if and only if, for some γ , $\langle \alpha, \langle \beta, \gamma \rangle \rangle$ belongs to UP_{1*} if and only if $\langle \alpha, \beta \rangle$ belongs to US_1^1 if and only if β belongs to $US_1^1 \upharpoonright \alpha$. Therefore X coincides with $US_1^1 \upharpoonright \alpha$. The proof of the second statement is similar. We leave the proof of the third statement to the reader.

(ii) Let X belong to Σ_1^1 and let Y be a closed subset of \mathcal{N} such that $X = \text{Ex}(Y)$ and let C be a decidable subset of \mathbb{N} such that, for every β , β belongs to Y if and only if, for each n , $\bar{\beta}n$ belongs to C . Observe that for every α , α belongs to X if and only if, for some β , for every n , $\overline{\langle \alpha, \beta \rangle}n$ belongs to C . Define a function f from \mathcal{N} to \mathcal{N} such that for every α, β, n , $(f| \alpha)(\bar{\beta}n) = 0$ if and only if $\overline{\langle \alpha, \beta \rangle}n$ belongs to C and observe

that f reduces X to E_1^1 . The proof of the second statement is similar. We leave the proof of the third statement to the reader.

(iii) Suppose that UP_1^1 belongs to Σ_1^1 and consider the *diagonal* set DP_1^1 consisting of all α such that $\langle \alpha, \alpha \rangle$ belongs to UP_1^1 . Also DP_1^1 belongs to Σ_1^1 . Find β such that DP_1^1 coincides with $US_1^1 \upharpoonright \beta$ and observe: $\langle \beta, \beta \rangle$ belongs to UP_1^1 if and only if $\langle \beta, \beta \rangle$ belongs to US_1^1 . Therefore $\langle \beta, \beta \rangle$ can not belong to either UP_1^1 or US_1^1 .

(iv) Let f be a function from \mathcal{N} to \mathcal{N} . We define α in \mathcal{C} as follows. For each a , $\alpha(a) = 1$ if and only if there exist initial segments b, c of a such that $f^b(c) > 1$ and for every proper initial segment d of c , $f^b(d) = 0$. Observe that for every β , for some n , $\alpha(\bar{\beta}n) = 1$ if and only if, for some n , $(f|\alpha)(\bar{\beta}n) > 0$. It follows that α belongs to A_1^1 if and only if $f|\alpha$ belongs to A_1^1 and: α belongs to E_1^1 if and only if $f|\alpha$ belongs to E_1^1 .

(v) Suppose that f is a function from \mathcal{N} to \mathcal{N} reducing A_1^1 to E_1^1 . Find α such that α belongs to A_1^1 if and only if $(f|\alpha)$ belongs to A_1^1 and: α belongs to E_1^1 if and only if $f|\alpha$ belongs to E_1^1 . We also know that α belongs to A_1^1 if and only if $f|\alpha$ belongs to E_1^1 . It follows that α cannot belong to either one of A_1^1 , E_1^1 . The proof of the second statement is similar. \square

9.12 Theorem: (*Intuitionistic Observations on the leading sets in Σ_1^1 and Π_1^1*)

- (i) For every function f from \mathcal{N} to \mathcal{N} mapping E_1^1 into A_1^1 there exists α such that both α and $f|\alpha$ belong to A_1^1 .
- (ii) The set E_1^1 is strictly analytical but not co-analytical, and the set A_1^1 is co-analytical but not strictly analytical.
- (iii) For every non-zero stump σ there exist functions γ, δ from \mathcal{N} to \mathcal{N} such that γ reduces A_σ to E_1^1 and maps E_σ into A_1^1 and δ reduces E_σ to E_1^1 and maps A_σ into A_1^1 .
- (iv) For every non-zero stump σ , for every positively Borel subset X of \mathcal{N} , for every function f from \mathcal{N} to \mathcal{N} mapping E_1^1 into X there exists α in A_1^1 such that $f|\alpha$ belongs to X .
- (v) For every non-zero stump σ , for every positively Borel subset X of \mathcal{N} for every function f from \mathcal{N} to \mathcal{N} mapping A_1^1 into X there exists α in E_1^1 such that $f|\alpha$ belongs to X .
- (vi) The sets E_1^1, A_1^1 are not positively Borel.

Proof: (i) Let f be a function from \mathcal{N} to \mathcal{N} mapping E_1^1 into A_1^1 . We define α in \mathcal{C} as follows. For each a , $\alpha(a) = 1$ if and only if there exist initial segments b, c of a such that $f^b(c) > 1$ and for every proper initial segment d of c , $f^b(d) = 0$. Observe that for every β , for some n , $\alpha(\bar{\beta}n) = 1$ if and only if for some n , $(f|\alpha)(\bar{\beta}n) \neq 0$. We claim that α belongs to A_1^1 . Let β belong to \mathcal{N} and consider the sequence γ such that for each n , $\gamma(\bar{\beta}n) = 0$ and for each a , if a is not an initial segment of β , then $\gamma(a) = \alpha(a)$. Observe that γ belongs to E_1^1 , therefore $f|\gamma$ belongs to A_1^1 and find n

such that $(f|\gamma)(\bar{\beta}n) \neq 0$. Also find m such that $f^{\bar{\beta}n}(\bar{\gamma}m) \neq 0$, and distinguish two cases. *Either* $\bar{\alpha}m = \bar{\gamma}m$, and therefore $(f|\alpha)(\bar{\beta}n) \neq 0$, and also for some k , $\alpha(\bar{\beta}k) = 1$, *or* $\bar{\alpha}m \neq \bar{\gamma}m$ and therefore, for some k , $\alpha(\bar{\beta}k) \neq \gamma(\bar{\beta}k)$ and $\alpha(\bar{\beta}k) \neq 0$.

(ii) We define a function f from \mathcal{N} to \mathcal{N} such that, for each α , for each n , $(f|\alpha)(\bar{\alpha}^1n) = 0$, and for each a , if a is not an initial segment of α^1 , then $(f|\alpha)(a) = \alpha^0(a)$. It is easy to see that E_1^1 coincides with the range of f . Therefore E_1^1 is strictly analytic. It follows from (i) that E_1^1 does not reduce to A_1^1 and so is not co-analytic. One may draw the same conclusion from Theorem 9.9(iv). It follows from Theorem 9.7(iv), the Boundedness Theorem, that the set A_1^1 is not strictly analytic.

(iii) We use induction on the set of stumps. We first define a function γ from \mathcal{N} to \mathcal{N} such that for each α , for each n , $(\gamma|\alpha)(\bar{0}n) = 0$ if and only if for each $j \leq n$, $\alpha(\langle j \rangle) = 0$, and, for each a , if a is not an initial segment of $\bar{0}n$, then $(\gamma|\alpha)(a) \neq 0$. Observe that γ simultaneously reduces A_{1*} to E_1^1 and E_{1*} to A_1^1 . We then define a function δ from \mathcal{N} to \mathcal{N} such that for each α , for each n , $(\delta|\alpha)(\langle n \rangle) \neq 0$ if and only if $\alpha(\langle n \rangle) = 0$, and, for each a , if $\text{length}(a) \neq 1$, then $(\delta|\alpha)(a) = 0$. Observe that δ simultaneously reduces E_{1*} to E_1^1 and A_{1*} to A_1^1 . Now assume that σ is a non-zero stump different from 1^* and that for each n there exist functions γ, δ from \mathcal{N} to \mathcal{N} such that γ reduces A_σ to E_1^1 and maps E_σ into A_1^1 and δ reduces E_σ to E_1^1 and maps A_σ into A_1^1 . Using the Second Axiom of Countable Choice we find γ, δ such that, for each n , γ^n, δ^n are functions from \mathcal{N} to \mathcal{N} , and γ^n reduces $A_{(\sigma^n)}$ to E_1^1 and maps $E_{(\sigma^n)}$ into A_1^1 and δ^n reduces $E_{(\sigma^n)}$ to E_1^1 and maps $A_{(\sigma^n)}$ into A_1^1 . We define a function f from \mathcal{N} to \mathcal{N} such that for all α , for all β , for all p , $(f|\alpha)(\bar{\beta}p) = 0$ if and only if for all n, j , if $\langle n \rangle * j \leq p$, then $(\delta^n|\alpha^n)(\bar{\beta}^nj) = 0$. Observe that $f|\alpha$ belongs to E_1^1 if and only if, for each n , $\delta^n|\alpha^n$ belongs to E_1^1 , therefore f reduces A_σ to E_1^1 . Observe also that for all α , if α belongs to E_σ , then for some n , $\delta^n|\alpha^n$ belongs to A_1^1 , and therefore $f|\alpha$ belongs to A_1^1 . We also define a function g from \mathcal{N} to \mathcal{N} such that, for all α , $(g|\alpha)(\langle \rangle) = 0$ and for all n, a , $(g|\alpha)(\langle n \rangle * a) = (\gamma^n|\alpha)(a)$. Observe that $g|\alpha$ belongs to E_1^1 if and only if, for some n , $\gamma^n|\alpha^n$ belongs to E_1^1 , therefore g reduces E_σ to E_1^1 . Observe also that for each α , if α belongs to A_σ , then for each n , $\gamma^n|\alpha^n$ belongs to A_1^1 and therefore $g|\alpha$ belongs to A_1^1 .

(iv) Let X be a positively Borel set and let f be a function from \mathcal{N} to \mathcal{N} mapping E_1^1 into X . Find a non-zero stump σ and a function g from \mathcal{N} to \mathcal{N} reducing X to A_σ . Also find a function h from \mathcal{N} to \mathcal{N} reducing E_σ to E_1^1 and mapping A_σ into A_1^1 . Observe that the composition of g, f and h maps E_σ into A_σ and, using the Borel Hierarchy Theorem, find α such that both α and $g|(f|(h|\alpha))$ belong to A_σ . Remark that $h|\alpha$ belongs to A_1^1 and $f|(h|\alpha)$ belongs to X .

(v) The proof is similar to the proof of (iv) and left to the reader.

(vi) is an easy consequence of (iv) and (v). In [17] the question how to prove that A_1^1 is not positively Borel was asked but not answered. \square

9.13 Theorem: (One Half of Souslin's Theorem)

- (i) For every stump σ , the set of all α such that α embeds into σ is a positively Borel subset of \mathcal{N} .
- (ii) For every subset X of \mathcal{N} , if X is both strictly analytic and co-analytic, then X is positively Borel.
- (iii) The set A_1^1 is not strictly analytic.

Proof: (i) This fact follows easily from the observation that for every non-zero stump σ , for every α , α embeds into σ if and only if for each m there exists n such that α^m embeds into σ^n .

(ii) Let f be a function from \mathcal{N} to \mathcal{N} such that X coincides with the range of f . Let g be a function from \mathcal{N} to \mathcal{N} reducing X to A_1^1 . Let h be the composition of g and f . Using the Boundedness Theorem, Theorem 9.7(iv), find β in A_1^1 such that, for every α , $h|\alpha$ embeds into β . Using Brouwer's Thesis, see Theorem 9.7(ii), we may assume that β is a stump. Observe that, for each α , α is positively Borel if and only if $g|\alpha$ embeds into β , therefore, by (i), X is positively Borel.

(iii) This follows from (ii) and Theorem 9.12(v). □

9.14 Theorem 9.13 is of limited application as every co-analytic subset of \mathcal{N} is perhaps and “most” positively Borel sets are not.

The distinction we have to make between analytic and strictly analytic is sometimes annoying. For instance, the set A_1^1 is not strictly analytic, but we would like to prove that the set A_1^1 is not analytic.

One may argue that A_1^1 , if analytic, must also be co-analytic, as follows. Suppose that there exists a closed subset X of \mathcal{N} such that $A_1^1 = Ex(X)$. It is reasonable to assume that the set X is given by a finite description. Therefore, for each a we may consider the question if a contains an element of X as a well-circumscribed proposition not involving incomplete objects. Using a so-called Brouwer-Kripke axiom we may form β in \mathcal{N} such that a contains an element of X if and only if for some n , $\beta(n) = 1$. Applying also the Second Axiom of Countable Choice we may form δ enumerating all numbers a such that a contains an element of X . But then, according to Theorem 9.5(ii), X is strictly analytic, and therefore, by Theorem 9.9(ix), also $A_1^1 = Ex(X)$ is strictly analytic. We then must conclude that A_1^1 is not analytic.

John Burgess, in [7], following [8], uses a term of Brouwer's and calls strictly analytic subsets of \mathcal{N} “*dressed spreads*”. Applying an unrestricted Brouwer-Kripke-axiom he proves that every analytic subset of \mathcal{N} is strictly analytic. The argument just given is essentially his.

9.15 Let X be a subset of \mathcal{N} . X is called an (*existential*) *projection of a co-analytic set* if and only if there exists a co-analytic subset Y of \mathcal{N} such that $X = Ex(Y)$. The class of these sets is denoted by Σ_2^1 . A subset X of \mathcal{N} is called a *universal projection of an analytic set* if and only if there exists an analytic subset Y of \mathcal{N} such that

$X = Un(Y)$. The class of these sets of denoted by $\mathbf{\Pi}_2^1$.

We define subsets US_2^1 and UP_2^1 of \mathcal{N} , as follows. US_2^1 is the set of all α such that, for some β , the sequence $\langle \alpha^0, \langle \alpha^1, \beta \rangle \rangle$ belongs to UP_1^1 , and UP_2^1 is the set of all α such that, for all β , the sequence $\langle \alpha^0, \langle \alpha^1, \beta \rangle \rangle$ belongs to US_1^1 . US_1^1 , UP_1^1 are called the *cataloguing* sets of $\mathbf{\Sigma}_2^1$, $\mathbf{\Pi}_2^1$, respectively.

We define subsets E_2^1 and A_2^1 of \mathcal{N} , as follows. E_2^1 is the set of all α such that for some β , for all γ there exists n , such that $\alpha(\langle \bar{\beta}n, \bar{\gamma}n \rangle) \neq 0$, and A_2^1 is the set of all α such that for all β there exists γ such that, for all n , $\alpha(\langle \bar{\beta}n, \bar{\gamma}n \rangle) = 0$. E_2^1 , A_2^1 are called the *leading* sets of $\mathbf{\Sigma}_2^1$, $\mathbf{\Pi}_2^1$, respectively.

9.16 Theorem: (*Properties of the classes $\mathbf{\Sigma}_2^1$ and $\mathbf{\Pi}_2^1$*).

- (i) For every subset X of \mathcal{N} , X belongs to $\mathbf{\Sigma}_2^1$ if and only if, for some α , X coincides with $US_2^1 \upharpoonright \alpha$ if and only if X reduces to E_2^1 .
- (ii) For every subset X of \mathcal{N} , X belongs to $\mathbf{\Pi}_2^1$ if and only if, for some α , X coincides with $UP_2^1 \upharpoonright \alpha$ if and only if X reduces to A_2^1 .
- (iii) Both $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ are subclasses of $\mathbf{\Sigma}_2^1$ as well as of $\mathbf{\Pi}_2^1$.
- (iv) For every sequence X_0, X_1, \dots of subsets of \mathcal{N} , if, for each n , X_n belongs to $\mathbf{\Sigma}_2^1$, then both $\bigcup_{n \in \mathbb{N}} X_n$ and $\bigcap_{n \in \mathbb{N}} X_n$ belong to $\mathbf{\Sigma}_2^1$.
- (v) (*Collapse of the Projective Hierarchy*)
For every subset X of \mathcal{N} , if X belongs to $\mathbf{\Sigma}_2^1$, then both $Ex(X)$ and $Un(X)$ belong to $\mathbf{\Sigma}_2^1$.
- (vi) $\mathbf{\Pi}_2^1$ is a subclass of $\mathbf{\Sigma}_2^1$.
- (vii) For every sequence X_0, X_1, \dots of subsets of \mathcal{N} , if, for each n , X_n belongs to $\mathbf{\Pi}_2^1$, then $\bigcap_{n \in \mathbb{N}} X_n$ belongs to $\mathbf{\Pi}_2^1$.
- (viii) For every subset X of \mathcal{N} , if X belongs to $\mathbf{\Pi}_2^1$, then $Un(X)$ belongs to $\mathbf{\Pi}_2^1$.

Proof: We only show part of (v) and leave the rest of the proof to the reader. Let X be a subset of \mathcal{N} belonging to $\mathbf{\Sigma}_2^1$. We want to prove that $Un(X)$ belongs to $\mathbf{\Sigma}_2^1$. Let Y be a subset of \mathcal{N} such that X coincides with $Ex(Y)$ and Y belongs to $\mathbf{\Pi}_1^1$. Let Z be an open subset of \mathcal{N} such that $Y = Un(Z)$.

Let C be a decidable subset of \mathbb{N} such that for every α , α belongs to Z if and only if, for some n , $\bar{\alpha}n$ belongs to C . Apply the Second Axiom of Continuous Choice and observe that for every α , α belongs to $Un(X)$ if and only if, for every β , $\langle \alpha, \beta \rangle$ belongs to X , if and only if for every β there exists γ such that $\langle \langle \alpha, \beta \rangle, \gamma \rangle$ belongs to Y , if and only if there exists ε in **Fun** such that $\varepsilon(0) = 0$ and for each β , $\langle \langle \alpha, \beta \rangle, \varepsilon|\beta \rangle$ belongs to Y . Then remark that for all ε , α , ε belongs to **Fun** and $\varepsilon(0) = 0$ and for each β , $\langle \langle \alpha, \beta \rangle, \varepsilon|\beta \rangle$ belongs to Y if and only if ε belongs to **Fun** and $\varepsilon(0) = 0$ and for all β , δ there exists n such that $\overline{\langle \langle \langle \alpha, \beta \rangle, \varepsilon|\beta \rangle, \delta \rangle}n$ belongs to C , if and only if ε belongs to **Fun** and $\varepsilon(0) = 0$ and for all $\beta, \delta, \zeta, \mu$ there exists n such that *either* $\overline{\langle \langle \langle \alpha, \beta \rangle, \zeta \rangle, \delta \rangle}n$

belongs to C or for some $i < n$, for some $j < \mu(n)$, $\varepsilon^i(\bar{\delta}j) \neq 0$, or, for some $i < n$, $\varepsilon^i(\bar{\delta}\mu(i)) \neq \zeta(i) + 1$. This shows that the set all pairs $\langle \varepsilon, \alpha \rangle$ such that ε belongs to **Fun** and $\varepsilon(0) = 0$ and for each β , $\langle \langle \alpha, \beta \rangle, \varepsilon|\beta \rangle$ belongs to Y , is a member of the class Π_1^1 , and that the set $Un(X)$ itself is a member of the class Σ_2^1 . \square

9.17 The previous Theorem shows that, in intuitionistic mathematics, Σ_2^1 is the class of all positive projective sets. One may ask if Π_2^1 is a proper subclass of Σ_2^1 and if the class Π_2^1 is closed under the operation of disjunction. We were unable to answer these questions.

9.18 Theorem: (*A Remnant of the Classical Projective Hierarchy Theorem*)

- (i) There exists γ belonging to neither one of US_2^1, UP_2^1 .
- (ii) For every function f from \mathcal{N} to \mathcal{N} there exists α such that α belongs to E_2^1 if and only if $f|\alpha$ belongs to E_2^1 and also: α belongs to A_2^1 if and only if $f|\alpha$ belongs to A_2^1 .
- (iii) There exists α belonging to neither one of E_2^1, A_2^1 .

Proof: (i) We let DP_2^1 be the set of all α such that $\langle \alpha, \alpha \rangle$ belongs to UP_2^1 . According to Theorem 9.15, DP_2^1 is a member of Σ_2^1 and we may find β such that DP_2^1 coincides with $US_2^1 \upharpoonright \beta$, therefore, for every α , $\langle \alpha, \alpha \rangle$ belongs to UP_2^1 if and only if $\langle \beta, \alpha \rangle$ belongs to US_2^1 . Define $\gamma := \langle \beta, \beta \rangle$ and observe that γ does not belong to either US_2^1 or UP_2^1 , as every element of US_2^1 is apart from every element of UP_2^1 .

(ii) Let f be a function from \mathcal{N} to \mathcal{N} . We construct α in \mathcal{C} in such a way that for all β, γ , there exists n such that $\alpha(\langle \bar{\beta}n, \bar{\gamma}n \rangle) = 1$ if and only if there exists n such that $(f|\alpha)(\langle \bar{\beta}n, \bar{\gamma}n \rangle) \neq 0$.

We define α by induction, as follows. Let n be a natural number and suppose we already decided about $\alpha(0), \dots, \alpha(n-1)$. We now say: $\alpha(n) = 1$ if and only if there exists b, c, p, q, j such that $\text{length}(b) = \text{length}(c)$ and $n = \langle b, c \rangle$ and $\text{length}(p) = \text{length}(q)$ and p, q are initial parts of b, c respectively and $j < n$ and $f^{\langle p, q \rangle}(\bar{\alpha}j) > 1$ and, for all $k < j$, $f^{\langle p, q \rangle}(\bar{\alpha}k) = 0$.

It is easy to see that α satisfies the requirements, and that α belongs to E_2^1 if and only if $f|\alpha$ belongs to E_2^1 and: α belongs to A_2^1 if and only if $f|\alpha$ belongs to A_2^1 .

(iii) Let f be a function from \mathcal{N} to \mathcal{N} reducing A_2^1 to E_2^1 and apply (ii), keeping in mind that every element of E_2^1 is apart from every element of A_2^1 . \square

9.19 The reader may compare Theorem 9.18 to our Remark 7.5.3.

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